DAE Approximations of PDE Modeled Control Problems

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Abstract Over the last decade there has been substantial progress on the development of theory and numerical methods for implicit systems of differential and algebraic equations (DAEs). In many control applications involving PDE models it is standard engineering practice to replace the PDEs by a finite dimensional system of ordinary differential equations. There are a variety of ways to do this approximation. Sometimes this approximation forms a DAE. While there has been a substantial amount of work on infinite dimensional control problems, there has been less attention paid to how the choice of approximation relates to the numerical and analytic properties of the finite dimensional DAE control system. In this paper we discuss some of the issues involved in this relationship.

Keywords: Differential Algebraic Equations, Partial Differential Equations, Index, Control Problems.

1 Introduction to DAEs

Many physical problems are most easily initially modeled as a nonlinear implicit system of differential and algebraic equations (DAE),

$$f(x', x, t) = 0 \tag{1}$$

with $f_{x'} = \partial f/\partial x'$ identically singular [1]. The index ν, which will be defined shortly, is one measure of how singular a DAE is. An ordinary differential equation is index zero, and increasing index implies more complex behavior. Constrained mechanical systems are frequently index two or three. Trajectory control problems which include the affects of actuator dynamics can be index six or more [2].

The obvious advantages in being able to work directly with the original DAE model include faster simulation because of reduced time needed for model manipulation, more detailed models, and exploitation of sparsity or other system structure. Over the last decade there has been considerable research on the numerical solution and analytic properties of DAEs. Reliable codes are becoming available for index one and some special classes of index two and three systems [1, 8]. New methods under development show the promise of working on even larger classes of problems [3].

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Suppose the DAE (1) is a system of n equations in the $(2n+1)$-dimensional variable (t, x, x') and that $f_{x'}$ is always singular. We also assume that f is sufficiently differentiable in the variables (t, x, x') so that all needed differentiations can be carried out. Intuitively, the DAE (1) is *solvable* in an open set $\Omega \subseteq \mathbf{R}^{2n+1}$ if the graphs $(t, x(t), x'(t))$ of the solutions x form a smooth $2m+1$ dimensional manifold in Ω and solutions are uniquely determined by their value x_0 at any t_0 such that $(t_0, x_0, v_0) \in \Omega$. More precise definitions appear in [2, 9]. Solvable DAEs are also sometimes called regular. Related references on DAEs can be found in [1, 5].

In general, the solution x of (1) is known to depend on derivatives of f. If (1) is differentiated k times with respect to t, we get the $(k+1)n$ *derivative array equations* [2]

$$F_k(x', w, x, t) = \begin{bmatrix} f(x', x, t) \\ \vdots \\ \dfrac{d^k}{dt^k}[f(x', x, t)] \end{bmatrix} = 0 \tag{2}$$

where $w = [x^{(2)}, \ldots, x^{(k+1)}]$.

The *index* ν of the DAE (1) is often taken to be the least integer k for which (2) uniquely determines x' for consistent (t, x). If such a k exists, then x' is a function of just (t, x) (for consistent (t, x)) so that $x' = g(x, t)$. For general unstructured DAEs, the index is actually a somewhat more subtle concept. The index given here is often called the *differentiation index* and denoted ν_d when more than one index is being considered. If $\nu > 1$, then the DAE is called *higher index*.

Consider the control problem

$$f(x', x, t) = B(x, t, u) \tag{3a}$$

$$y = g(x, t, u) \tag{3b}$$

The first equation (3a) describes a physical system with u being a control variable. The equation (3b) is viewed in two distinct ways. y could represent an output (observation) that is available for designing a control u with some desired behavior. Alternatively, and the interpretation we shall mostly pursue, (3b) describes desired behavior. In this setting, y is a known function and (3) is a DAE in the unknowns (x, u). The requirement (3b) is then sometimes called a prescribed path constraint or a program constraint. These constraints should be distinguished from physical or material constraints which arise from the physical plant's design. We include physical constraints in (3a).

A closely related concept to the index is the relative degree. To simplify our discussion of the relative degree, suppose for the moment that (3) is

$$x' = h(x,t) + B(x,t)u \quad (4a)$$
$$y = g(x,t) \quad (4b)$$

Differentiating (4b) with respect to t and using (4a) gives

$$y' = g_x x' + g_t = g_x(x,t)[h(x,t) + B(x,t)u] + g_t(x,t) \quad (5)$$

If $g_x(x,t)B(x,t)$ is full column rank, we can solve for u. Suppose that this is not the case and for simplicity also assume that $g_x(x,t)B(x,t) = 0$. Then (5) can be written as

$$y' = g_1(x,t) \quad (6)$$

Replacing (4b) with (6) we repeat the process until we can solve for u. Algorithms for the case when $(g_i)_x B \neq 0$ but $(g_i)_x B$ is also not full column rank for some i are discussed in the literature. The *relative degree* is the number of times that the output equation (4b) has to be differentiated in order to determine the control. If the physical process is modeled by an ordinary differential equation as in (3a), then assuming that both concepts are well defined, the relative degree of (4) is one less than the index of (4) with respect to (x,u). However, for implicitly modeled processes the index could be much higher than the relative degree. The relative degree talks only about those parts of the control u and the state x that directly affect the output y. There can be other dynamics present when $y \equiv 0$, called *zero dynamics*, which can also affect the index [2]. The relative degree plays a fundamental role in the design of path following controllers in nonlinear control theory.

Given a system modeled with PDEs, such as a robotic arm with some flexible links, it is standard engineering practice to approximate this infinite dimensional system with a finite dimensional one. There are a variety of ways to do this including finite elements and modal approximations. Once the finite dimensional model has been constructed the scientist may attempt to design controllers and perform simulations. In the case of complex systems, initial evaluation of the control design is often done using numerical simulations. In some cases, the control itself is computed at least in part utilizing a numerical integrator [6].

There has been a considerable amount of fundamental research relating the control computed for the finite dimensional problem and the infinite dimensional problem. This is especially true for LQR approaches where the control is being chosen to minimize some type of cost criteria. There has also been some research on path following controllers based on modern nonlinear control theory, see [11, 12] for example. Much of the analytical research to date has understandably focused on systems with few flexible links. With more complex models it will be desirable to utilize implicit models and rely on simulation in a fundamental way. There has been relatively little discussion about the effect of the chosen approximation process on the structure of the resulting DAE and the effect this has on numerical simulation and control design.

This paper begins an examination of the properties of the DAEs which arise in the finite dimensional approximations of infinite dimensional systems and the consequences of these DAE properties for control and simulation. In this paper we present some examples and discuss some of the behavior of interest to us rather than give technical results. DAEs can arise either because of the original control problem or during the approximation process. We shall give examples of both. In Section 2 we give a heat control example that shows how the relative degree and the index can vary with the type of approximation. For this problem the infinite dimensional formulation is also implicit. In Section 3 we shall briefly mention some of what is known for flexible mechanical systems in this context and discuss what this could mean for nonlinear control and numerical simulation. In Section 4 we discuss the computation of approximate inertial manifolds. Here the original problem is not implicit but the approximation may be chosen to be a DAE.

2 Simple Heat Example

Suppose that we have the usual heat equation in a horizontal insulated homogeneous rod of length π whose right endpoint is held at zero and whose initial temperature profile is $T(x)$.

$$u_t = c u_{xx} \quad (7a)$$
$$u(x,0) = T(x), u(\pi,t) = 0, \quad t \geq 0, \ 0 \leq x \leq \pi \quad (7b)$$

The control problem is to find a left boundary condition $\phi(t)$ so that the temperature at a fixed interior point p satisfies a given smooth temperature profile $P(t)$. That is, the solution must satisfy the left boundary condition

$$u(0,t) = \phi(t) \quad (8)$$

and the desired constraint,

$$u(p,t) = P(t) \quad (9)$$

which in keeping with the finite dimensional case we call a path constraint. Thus ϕ is the control. We take $c = 1$ and $p = \pi/2$. Our interest is in the various types of DAEs that arise and not on how to best solve this problem.

2.1 Continuous Solution

The analytic solution of (7),(8) can be found using the usual Fourier series techniques. To simplify the discussion we assume in this section that the initial temperature profile is zero and that $P(0) = 0$ Let

$$\frac{\pi - x}{\pi} = \sum_{n=1}^{\infty} \beta_n \sin nx, \quad \beta_n = \frac{2}{n\pi} \quad (10)$$

and f_n be the solution of

$$f_n'(t) = -n^2 f_n(t) - \phi'(t)\beta_n, \quad f_n(0) = 0 \qquad (11)$$

Then

$$u(x,t) = \phi(t)\frac{\pi - x}{\pi} + \sum_{n=1}^{\infty} f_n(t)\sin nx - \phi(0)\sum_{n=1}^{\infty}\beta_n e^{-n^2 t}\sin nx \qquad (12)$$

The path constraint (9) is

$$P(t) = u(\frac{\pi}{2}, t) = \phi(t)\frac{1}{2} + \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi}{2} - \phi(0)\sum_{n=1}^{\infty} e^{-n^2 t}\beta_n \sin\frac{n\pi}{2} \qquad (13)$$

Now from (11)

$$f_n(t) = -\int_0^t e^{-n^2(t-s)}\phi'(s)\beta_n ds = -\beta_n\phi(t) + n^2\beta_n\int_0^t e^{-n^2(t-s)}\phi(s)ds + \phi(0)\beta_n e^{-n^2 t} \qquad (14)$$

Thus

$$P(t) = \sum_{n=1}^{\infty} n^2\beta_n\left[\int_0^t e^{-n^2(t-s)}\phi(s)ds\right]\sin\frac{n\pi}{2} \qquad (15)$$

Define

$$\Theta(z) = \sum_{n=1}^{\infty}\frac{2n}{\pi}e^{-n^2 z}\sin\frac{n\pi}{2} \qquad (16)$$

Then the relationship (15) becomes

$$P(t) = \int_0^t \Theta(t-s)\phi(s)ds \qquad (17)$$

Note, however, that $\Theta(0)$ is not defined since the series (16) is divergent at $z = 0$. Thus it might be preferable to consider instead the relationship

$$P(t) = \frac{1}{2}\phi(t) - \sum_{n=1}^{\infty}\frac{2}{n\pi}\int_0^t e^{-n^2(t-s)}\phi'(s)ds\sin\frac{n\pi}{2} - \phi(0)\sum_{n=1}^{\infty} e^{-n^2 t}\frac{2}{n\pi}\sin\frac{n\pi}{2} \qquad (18)$$

which is conditionally convergent for $t = 0$ to $P(0) = 0$.

In any event, given P, it is not trivial to recover the desired control ϕ. This control problem is known to be ill conditioned but it serves to easily illustrate several interesting phenomena.

2.2 Finite System: MOL

As our first approximation we shall apply a standard method of lines (MOL) approach. That is we shall discretize the spatial variable to get a system of ordinary differential equations. Take a partition $x_i,\ i = 0, \ldots, N+1$ of the interval $[0\ \pi]$ with N an odd integer. We shall assume equal spacing h. Let $K = (N+1)/2$. Thus $x_K = \pi/2$. Let $u_i(t) = u(x_i, t)$. We approximate the u_{xx} term in (7) with the standard second difference $h^{-2}(u_{i+1} - 2u_i + u_{i-1})$ to get the following system of DAEs. A different approximation would produce a DAE with similar structure. Here we have used the boundary condition $u_{N+1} = 0$ and the desired path constraint $u_K = P(t)$.

$$\begin{aligned}
h^2 u_1' &= -2u_1 + u_2 + u_0 && (19a)\\
h^2 u_2' &= u_1 - 2u_2 + u_3 && (19b)\\
\vdots \quad & \quad \vdots && (19c)\\
h^2 u_{K-1}' &= u_{K-2} - 2u_{K-1} + P(t) && (19d)\\
h^2 P'(t) &= u_{K+1} + u_{K-1} - 2P(t) && (19e)\\
h^2 u_{K+1}' &= -2u_{K+1} + cu_{K+2} + P(t) && (19f)\\
\vdots \quad &= \quad \vdots && (19g)\\
h^2 u_N' &= u_{N-1} - 2u_N && (19h)
\end{aligned}$$

Thus the system (19) is a cascade of two systems. Let $\overline{u} = [u_0, \ldots, u_{K-1}]^T$, $\overline{v} = [u_{K+1}, \ldots, u_N]^T$. Then the system (19) can be written as

$$\begin{aligned}
h^2 E\overline{u}' &= H\overline{u} + J\overline{v} + b(t) && (20a)\\
h^2\overline{v}' &= M\overline{v} + r(t) && (20b)
\end{aligned}$$

where

$$E = \begin{bmatrix} 0 & 1 & 0 & \cdot & 0 \\ \cdot & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 & 1 \\ 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & -2 & 1 & \cdot & 0 \\ 0 & 1 & -2 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 & -2 \\ \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} -2 & 1 & 0 & \cdot & \cdot \\ 1 & -2 & 1 & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & 1 & -2 & 1 \\ \cdot & \cdot & 0 & 1 & -2 \end{bmatrix}, J = \begin{bmatrix} 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 0 \end{bmatrix},$$

$$b(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ P(t) \\ -2P(t) - h^2 P'(t) \end{bmatrix}, \ r(t) = \begin{bmatrix} P(t) \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

The system (20b) is an ODE that can be solved for $\overline{v}$ and in particular for u_{K+1}. Given u_{K+1}, the DAE (20a) is a Hessenberg system of size K [1]. Hence the DAE has index K in $\overline{u}$ or index $K+1$ in $(\overline{u}, u_K)$. In this example the index and relative degree go to infinity as $h \to 0^+$.

What happens if we apply a numerical method like BDF to this DAE? Because of the way the variables propagate we see that u_{K-1} will be accurate after one time step. However, the value of u_{K-2} will take two time steps to become accurate. Thus there is a triangular shaped region of nonconvergence. At u_0 the region will be $K+1$ time steps long.

Suppose for simplicity that we were using a backward Euler method for the time integration. If we wanted the same accuracy in the time direction as the spatial direction we could take the time step to be $O(h^2)$. Since $K = O(h^{-1})$ we see that the length of the boundary layer in computing $u_0 = \phi$ is $O(h)$. Of course, since the answer involves repeated numerical differentiations there is a question of how practical this approach is. In fact, for very high index it is probably only of theoretical interest. Also, given a solution of an approximation it is crucial to determine in what sense, if any, it approximates the solution of the infinite dimensional problem. However, this will not be addressed in this paper.

2.3 Finite System: Modal

In this section we shall examine what happens if we use a modal approach for approximating the control problem (7), (8), (9). That is, we shall project the problem onto the span of the first N eigenfunctions. Utilizing the calculations in Section 2.1 we take

$$u = \frac{\pi - x}{\pi}\phi(t) + \sum_{n=1}^{\infty} u_n(t)\sin nx \tag{21}$$

where the u_n are unknown functions. We substitute (21) into the PDE, initial and boundary conditions, and truncate at N terms. There is some leeway in how to do this and an alternative will be discussed shortly. But for now we shall consider the following problem

$$\begin{aligned} u_k'(t) = -k^2 u_k(t) - \beta_k \phi'(t), \quad 1 \le k \le N \quad && (22a)\\ \phi(t)\sum_{k=1}^{N}\beta_k \sin\frac{k\pi}{2} + \sum_{k=1}^{N} u_k(t)\sin\frac{k\pi}{2} \;=\; P(t) \quad && (22b)\\ u_k(0) = T_k - \beta_k\phi(0), \qquad k \text{ odd} \quad && (22c)\\ u_k(0) = T_k, \qquad k \text{ even} \quad && (22d) \end{aligned}$$

where (22b) is the path constraint (9) and $T(x) = \sum_{k=1}^{\infty} T_k \sin nx$.

This is a DAE in the variables $\{u_k, \phi\}$. We shall analyze (22) carefully. We introduce the following notation for $1 \le k \le N$. Let $\overline{u} = [u_1, u_3, \ldots]$ be the u_k for odd k. Let $\overline{v}$ be the u_k for even k. Let D be a diagonal matrix with diagonal the squares of odd integers in increasing order and E a diagonal matrix with diagonal the squares of even integers in increasing order starting with 2. Similarly let b be a column vector of the β_k for odd k values and d the values of β_k for even k values. Finally let m be a row vector $[1, -1, 1, \ldots]$. Then the modal approximation (22) is

$$\begin{aligned} \overline{u}' + b\phi' &= -D\overline{u} && (23a)\\ (mb)\phi + m\overline{u} &= P(t) && (23b)\\ \overline{v}' + d\phi' &= -E\overline{v} && (23c) \end{aligned}$$

Since P is assumed known, the equations (23) form a DAE in $\{\overline{u}, \overline{v}, \phi\}$. The first two equations form a DAE in just $\{\overline{u}, \phi\}$. Once this DAE is solved, $\overline{v}$ is given by the ODE (23c). We shall focus on the DAE in (23a), (23b);

$$\begin{bmatrix} I & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \overline{u} \\ \phi \end{bmatrix}' = \begin{bmatrix} -D & 0 \\ m & mb \end{bmatrix}\begin{bmatrix} \overline{u} \\ \phi \end{bmatrix} - \begin{bmatrix} 0 \\ P \end{bmatrix} \tag{24}$$

We examine the matrix pencil for (24) using the standard reduction procedure consisting of differentiating constraints and then performing row operations. Differentiating the second line of (24), eliminating the (2,1)-block coefficient of the derivative terms, and then differentiating the new constraint gives

$$\left\{\begin{bmatrix} I & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -D & 0 \\ m & mb \end{bmatrix}\right\} \rightarrow \left\{\begin{bmatrix} I & b \\ mD & 0 \end{bmatrix}, \begin{bmatrix} -D & 0 \\ 0 & 0 \end{bmatrix}\right\}$$

But $\begin{bmatrix} I & b \\ mD & 0 \end{bmatrix}$ is invertible since $mDb \neq 0$. Thus we get that the DAE (24) is index two independent of the order N of the modal approximation.

Note that $\lim_{N\to\infty} mb = \frac{1}{2}$. A slightly different modal approximation is gotten by using the actual sum of the series multiplying ϕ in the constraint rather than the approximation used in (22b). That is, we let $x = \pi/2$ in (21) and then truncate the series. This gives the system

$$\begin{aligned} \overline{u}' + b\phi' &= -D\overline{u} && (25a)\\ (1/2)\phi + m\overline{u} &= P(t) && (25b)\\ \overline{v}' + d\phi' &= -E\overline{v} && (25c) \end{aligned}$$

instead of (23). Equation (25) also forms a DAE in $\{\overline{u}, \overline{v}, \phi\}$ which appears to have a similar structure to (23). In matrix form the DAE in $\{\overline{u}, \phi\}$ given by (25) is now

$$\begin{bmatrix} I & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \overline{u} \\ \phi \end{bmatrix}' = \begin{bmatrix} -D & 0 \\ m & 1/2 \end{bmatrix}\begin{bmatrix} \overline{u} \\ \phi \end{bmatrix} - \begin{bmatrix} 0 \\ P \end{bmatrix} \tag{26}$$

We again examine the matrix pencil for (26) using the reduction procedure.

$$\left\{\begin{bmatrix} I & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -D & 0 \\ m & 1/2 \end{bmatrix}\right\} \rightarrow \left\{\begin{bmatrix} I & b \\ 0 & \delta \end{bmatrix}, \begin{bmatrix} -D & 0 \\ mD & 0 \end{bmatrix}\right\}$$

where $\delta = \frac{1}{2} - mb$. But $\begin{bmatrix} I & b \\ 0 & \delta \end{bmatrix}$ is invertible since $\delta \neq 0$. Thus we get that the DAE (26) is index one independent of the order N of the modal approximation. Note, however, that as N increases we get $\delta \to 0$ so that the index one system becomes increasingly singular. In similar situations elsewhere, it usually follows that the computed solutions approach those of the higher index system and the numerical behavior is that of the higher index system. Thus it is sometimes better to just use the index two formulation and avoid some of the conditioning problems.

To try and get a better feel for what is happening here let us look at the infinite dimensional version of the equations. Let a hat over any term denote a similar object but constructed from the terms discarded in making the modal approximation. Then we have that (22) is

$$\left[\begin{array}{ccc} I & 0 & b \\ 0 & I & \widehat{b} \\ 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \widehat{\overline{u}} \\ \phi \end{array}\right]' = \left[\begin{array}{ccc} -D & 0 & 0 \\ 0 & -\widehat{D} & 0 \\ m & \widehat{m} & 1/2 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \widehat{\overline{u}} \\ \phi \end{array}\right] - \left[\begin{array}{c} 0 \\ 0 \\ P \end{array}\right] \quad (27)$$

where $\widehat{\overline{u}}$ is an infinite dimensional variable.

Differentiating the bottom equation in (27), performing one elimination, and then differentiating the constraint again gives

$$\left[\begin{array}{ccc} I & 0 & b \\ 0 & I & \widehat{b} \\ mD & \widehat{m}\widehat{D} & 0 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \widehat{\overline{u}} \\ \phi \end{array}\right]' = \left[\begin{array}{ccc} -D & 0 & 0 \\ 0 & -\widehat{D} & 0 \\ 0 & 0 & 0 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \widehat{\overline{u}} \\ \phi \end{array}\right] - \left[\begin{array}{c} 0 \\ 0 \\ P'' \end{array}\right] \quad (28)$$

However, the matrix version of the algorithm now runs into trouble. We can eliminate the mD term in the first matrix in (28) as before but we cannot eliminate the (3,2) term $\widehat{m}\widehat{D}$ since $\widehat{m}\widehat{D}\widehat{b}$ is not a convergent series.

2.4 Dynamics of the Two Modal Equations

Given a finite dimensional approximation, its dynamical behavior is important both in designing controls and in carrying out numerical simulations. In this section we shall briefly examine the dynamics of the two modal DAE approximations in $\{\overline{u}, \phi\}$. We consider first

$$\left[\begin{array}{cc} I & b \\ mD & 0 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \phi \end{array}\right]' = \left[\begin{array}{cc} -D & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \phi \end{array}\right] - \left[\begin{array}{c} 0 \\ P'' \end{array}\right] \quad (29)$$

computed from the index two formulation (24). Inverting the leading coefficient we get the ODE

$$\left[\begin{array}{c} \overline{u} \\ \phi \end{array}\right]' = \left[\begin{array}{cc} -D + \gamma bmD^2 & 0 \\ -\gamma mD^2 & 0 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \phi \end{array}\right] - \left[\begin{array}{c} \gamma bP'' \\ -\gamma P'' \end{array}\right] \quad (30a)$$

where $\gamma = (mDb)^{-1}$ along with the two scalar constraints

$$m\overline{u}(t) + (mb)\phi(t) = P \quad (30b)$$

$$mD\overline{u}(t) = -P' \quad (30c)$$

Note that if we multiply the first equation of (30a) by mD we get that $mD\overline{u}' = -P''$. Since this is the first derivative of (30c) we have that (30c) is an invariant for (30a). It will correspond to a zero eigenvalue in the coefficient matrix in (30a). Thus we have the system

$$\overline{u}' = [-D + \gamma bmD^2]\overline{u} - \gamma bP'' \quad (31a)$$

$$\phi(t) = (mb)^{-1}[P(t) - m\overline{u}] \quad (31b)$$

The nonzero eigenvalues of $-D + \gamma bmD^2$ will be identical to those of the index two DAE.

On the other hand, the index one system (26) leads to

$$\left[\begin{array}{cc} I & b \\ m & 1/2 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \phi \end{array}\right]' = \left[\begin{array}{cc} -D & 0 \\ 0 & 0 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \phi \end{array}\right] + \left[\begin{array}{c} 0 \\ P' \end{array}\right] \quad (32)$$

Inverting the leading coefficient we get the ODE

$$\left[\begin{array}{c} \overline{u} \\ \phi \end{array}\right]' = \left[\begin{array}{cc} -D - \delta^{-1}bmD & 0 \\ \delta^{-1}mD & 0 \end{array}\right]\left[\begin{array}{c} \overline{u} \\ \phi \end{array}\right] - \left[\begin{array}{c} \delta^{-1}bP' \\ -\delta^{-1}P' \end{array}\right] \quad (33a)$$

along with the scalar constraint

$$m\overline{u}(t) + (mb)\phi(t) = P(t) \quad (33b)$$

Thus we have

$$\overline{u}' = [-D - \delta^{-1}bmD]\overline{u} - \delta^{-1}bP' \quad (34a)$$

$$\phi(t) = (mb)^{-1}[P(t) - m\overline{u}] \quad (34b)$$

The eigenvalues of $-D - \delta^{-1}bmD$ will be those of the index one system. The eigenvalues of $-D + \gamma bmD^2$ and $-D - \delta^{-1}bmD$ for $N = 6$ are

Index One	Index Two
8.8306	0.0000
-3.4579 +15.3064i	-1.0000 + 8.0000i
-3.4579 -15.3064i	-1.0000 - 8.0000i

Eigenvalues with $N = 6$.

Note that the index one problem is unstable whereas the index two system is stable. The $N = 10$ case is more typical. The index one problem has the additional large real eigenvalue 12.9763. The remaining eigenvalues are comparable. There are also eigenvalues with positive real part of large modulus but the real part is much smaller. These eigenvalues can create difficulties for numerical integrators even if they do not pose problems in terms of the size of solutions.

Index One	Index Two
12.9763	0.0000
-39.2325 +32.6794i	-34.6453 +24.5866i
-39.2325 -32.6794i	-34.6453 -24.5866i
0.7265 +20.4477i	0.6453 + 9.0669i
0.7265 -20.4477i	0.6453 - 9.0669i

Eigenvalues with $N = 10$.

Looking at these problems we would expect that as N increases that the slowly increasing instability of the index two system should cause the solution to become more erratic with larger t.

2.5 Summary of The Heat Example

We have taken a boundary control problem and examined the structure of three finite dimensional approximations.

The MOL finite difference approximation led to a DAE whose index went to infinity as the spatial mesh went to zero. Two modal approximations were considered. The index (relative degree) was independent of the order of the approximation. One approximation was index two and one was index one. The index one approximation had a positive real eigenvalue of largest real part that the index two problem did not have so that the index one problem was more unstable than the index two problem. Whether this type of behavior will be found in other types of problems remains to be determined. This example also illustrated that the finite dimensional DAE can be easy to solve numerically even if the infinite dimensional control problem is difficult. Also, the relative degree of the approximation can remain constant even if the infinite dimensional control problem is ill-conditioned.

3 Mechanical Systems

Most of the literature on nonlinear flexible mechanical systems, for example in robotics, concerns systems with few flexible links. In addition, the arms are usually nonredundant. Suppose, as in [12], that we have a simple robotic arm with a finite number of links, some of which are flexible, and that we are interested in path control. Assuming a modal approximation for the flexible links we get a DAE. The relationship between the type of modal system, choice of approximating and simplifying assumptions, and the resulting DAE properties is a topic of current interest.

In many situations what happens is that the standard nonlinear control theory results in tracking the desired trajectory in theory but that there are nontrivial zero dynamics. The zero dynamics often correspond to the flexible modes and may be highly oscillatory. This creates difficulty for both control and numerical simulation.

Even if the flexible modes are lightly damped, we can get a DAE with eigenvalues with large imaginary part, but small negative real part. If one is using an implicit integrator for the simulations, this "stiffly oscillatory" behavior can create numerical difficulties unless the simulation code is specifically designed to deal with high frequency lightly damped oscillations. The design of robust numerical methods for such systems is a topic of current interest in the numerical analysis community.

The situation is even more interesting in the case of redundant manipulators. For these systems the zero dynamics can contain not only oscillatory but also other mechanical dynamics. Here the original system is also a DAE. One can easily construct theoretical control problems where there are constraints on the zero dynamics which cause the index of the DAE to be more than one higher than the relative degree. Whether this can happen in complex flexible system approximations remains to be seen. However, the presence of such higher index zero dynamics could lead to a failure of most current numerical simulation packages and difficulty in control design.

4 Approximate Inertial Manifolds

For our discussion we can summarize the AIM approach as follows. Details on approximate inertial manifolds for PDEs can be found in [4, 7, 10].

Suppose that we have a differential equation

$$\frac{du}{dt} + Au + F(u) = 0 \tag{35}$$

and A is a self-adjoint linear transformation. Eventually we shall assume u to be in an infinite dimensional space and A a differential operator. Let P be a self-adjoint projection that commutes with A and let $Q = I - P$. Define $p = Pu, q = Qu$. Then (35) may be written as

$$p' = -A_1 p + PF(p+q) \tag{36a}$$
$$q' = -A_2 q + QF(p+q) \tag{36b}$$

Suppose now that the eigenvalues of A_2 are larger then those of A_1. Then if we rewrite (36) as

$$p' = -A_1 p + PF(p+q) \tag{37a}$$
$$A_2^{-1} q' = -q + A_2^{-1} QF(p+q) \tag{37b}$$

we have what appears to be a singular perturbations problem. Since A_2^{-1} is small the reduced system we are led to is the DAE

$$p' = -A_1 p + PF(p+q) \tag{38a}$$
$$0 = -q + A_2^{-1} QF(p+q) \tag{38b}$$

There always remains the question of how good an approximation (38) is. In the work on inertial manifolds the situation is as follows. We assume that (35) is a dissipative partial differential equation. For our purposes this includes the assumption that A is nonsingular. P is the projection onto the sum of the m eigenvectors of A corresponding to the smallest m eigenvalues. (Recall $\lambda_i \to \infty$).

In this situation it is sometimes the case that all solutions of (35) approach a finite dimensional manifold $\mathcal{M}_\mathcal{I}$ which includes the limit set. Thus much of the qualitative behavior of the PDE can be determined by studying the qualitative behavior of the differential equation restricted to $\mathcal{M}_\mathcal{I}$. One would expect that for large enough m that $P\mathcal{M}_\mathcal{I}$ would have dimension $\dim(\mathcal{M}_\mathcal{I})$. Clearly this happens, at least locally.

Another common assumption is that $\mathcal{M}_\mathcal{I}$ may be given as the graph of a function

$$q = \Phi(p) \tag{39}$$

Note that (39) can only hold if $\dim(p) = \dim(\mathcal{M}_\mathcal{I})$ which requires knowledge of $\mathcal{M}_\mathcal{I}$. This is sometimes available.

Given that the manifold can be described by (39), the behavior of the PDE restricted to the manifold $\mathcal{M}_\mathcal{I}$ is completely described by the ODE

$$p' = -A_1 p + PF(p + \Phi(p)) \tag{40}$$

Analytically there are two separate issues. One is showing that there is an inertial manifold for a given equation. However, even if one cannot show there is an inertial manifold for the PDE, if there is eigenvalue separation between A_1 and A_2, then one can sometimes argue that the solutions of (38) still capture some of the behavior of the PDE.

If there is an inertial manifold, then computing it exactly is not usually possible so it is necessary to approximate it. This again leads us to the DAE (38). To generate Φ one uses approximate solutions of (38b). Suppose we rewrite (38b) as a fixed point problem.

$$q = A_2^{-1} QF(p+q) \tag{41}$$

We consider p as given. Then the $(n+1)$-th approximant is given by

$$\Phi_{n+1}(p) = A_2^{-1} QF(p + \Phi_n(p)) \tag{42}$$

Usually one takes $\Phi_0 = 0$ so that Φ_0 gives the usual linear (Galerkin) approximation. Then (42) would give for $n = 0, 1$,

$$\begin{aligned} \Phi_1(p) &= A_2^{-1} QF(p) && (43a) \\ \Phi_2(p) &= A_2^{-1} QF(p + A_2^{-1} QF(p)) && (43b) \end{aligned}$$

The hope is that the Φ_n arising from solving (41) are related to the Φ in (39) which describes the manifold $\mathcal{M}_{\mathcal{I}}$. This can be proven in some circumstances.

4.1 What Can DAEs Do for AIMs?

The DAE (38) is somewhat unusual with respect to the standard DAE theory since p is a finite dimensional vector and q is *infinite* dimensional. This opens the door for certain types of behavior that have not been adequately studied. One consequence, is that the relation (38b) must be solved in an approximate manner.

The first, and most obvious point, is that the AIM problems are always in the form of (38). We shall assume for our discussion that there is in fact an AIM, which we denote $\mathcal{M}_{\mathcal{I}}$, of dimension $d_{\mathcal{M}}$, and discuss the approximation problem. It will be important to keep in mind that while $\mathcal{M}_{\mathcal{I}}$ is assumed finite dimensional, it is not assumed that it lies in any finite dimensional subspace much less the one spanned by the first m eigenvectors of A.

Note that $d_{\mathcal{M}}$ is not necessarily known. The choice of m is motivated by the eigenvalue separation of A and the desire to keep m as small as possible for computational reasons. If we take too low a dimension, then the ODE does not capture the dynamics of the solution manifold. On the other hand, we will get stiffness of the ODE, and hence increased computational complexity, if we take too large an m.

Finally, note that the contraction argument for (41) essentially means that $I - A_2^{-1} QF_q$ is nonsingular. Thus the problem is index one, although in a nontrivial infinite dimensional sort of way.

The study of inertial manifolds poses several questions for study in the context of DAEs, particularly with infinite dimensional systems. What can the study of DAEs contribute to the complex, but special, problems that arise in AIM? Examples are reported in the literature where the AIM approach does not work well because the fast variables q in (36) have an important effect on the slow variables p. Also the use of Φ_i for small i may not fully capture this effect.

One thing that the DAE theory can do is to change ones perspective. In studying a problem the goal is no longer to change a DAE into an ODE but rather into a *nicer DAE*. The following scenario may be helpful.

We make the same assumptions as before on A but this time we introduce two eigenspace projections P_1, P_2 with ranks m_1, m_2. The first projection is the same as P_m. The second includes extra modes that we suspect may be important in the dynamics. A similar approach is used for flexible mechanical systems in [12]. Thus instead of (36) we have

$$\begin{aligned} p_1' &= -A_1 p_1 + P_1 F(p_1 + p_2 + q) && (44a) \\ p_2' &= -A_2 p_2 + P_2 F(p_1 + p_2 + q) && (44b) \\ q' &= -A_3 q + QF(p_1 + p_2 + q) && (44c) \end{aligned}$$

where p_1 and p_2 are finite dimensional.

Suppose now that the eigenvalues of A_2 and A_3 are larger than those of A_1. The eigenvalues of A_2 are not necessarily smaller than those of A_3. This leads to the consideration of the reduced order problem.

$$\begin{aligned} p_1' &= -A_1 p_1 + P_1 F(p_1 + p_2 + q) && (45a) \\ 0 &= -A_2 p_2 + P_2 F(p_1 + p_2 + q) && (45b) \\ 0 &= -A_3 q + QF(p_1 + p_2 + q) && (45c) \end{aligned}$$

So far this looks just like the standard approach with q replaced by q, p_2 and p replaced by p_1. However, only q is an infinite dimensional variable. We shall use an iterative method to solve (45c) for q, but leave (45a), (45b) as a nonlinear finite dimensional DAE which can be integrated with existing codes. That is, we solve

$$\begin{aligned} p_1' &= -A_1 p_1 + P_1 F(p_1 + p_2 + \overline{\Phi}(p)) && (46a) \\ 0 &= -A_2 p_2 + P_2 F(p_1 + p_2 + \overline{\Phi}(p)) && (46b) \end{aligned}$$

where (45c) is solved to give $q = \overline{\Phi}(p)$. For example, suppose that one were to use the Galerkin approximations for solving the infinite dimensional nonlinear equations. The usual approach applied to (45) would be to solve the ODE

$$p_1' = -A_1 p_1 + P_1 F(p_1) \tag{47}$$

gotten by setting $p_2 = 0,\ q = 0$ in (45) We are suggesting instead the solution of the DAE

$$\begin{aligned} p_1' &= -A_1 p_1 + P_1 F(p_1 + p_2) && (48a) \\ 0 &= -A_2 p_2 + P_2 F(p_1 + p_2) && (48b) \end{aligned}$$

gotten by setting $q = 0$.

To illustrate how this can sometimes make a difference, we shall consider a simple linear example where some of the fast dynamics are strongly coupled with the slow dynamics.

Example 1 Consider the DAE

$$x' = -x + y + z \quad (49a)$$

$$y' = -10y + 20x + z \quad (49b)$$

$$z' = -20z \quad (49c)$$

The eigenvalues of (49) are $\{0.8443, -11.8443, -20\}$. Thus the dynamics on the 1-dimensional invariant manifold are

$$w' = 0.8443w \quad (50)$$

In the notation of (44) we take $x = p_1,\ y = p_2,\ z = q$. If we apply the approximation (47) we get

$$x' = -x \quad (51a)$$

$$y = 0 \quad (51b)$$

$$z = 0 \quad (51c)$$

which has the wrong stability behavior in (51a) for x. However, if we use the reduction to a DAE as in (48) we get the DAE

$$x' = -x + y \quad (52a)$$

$$0 = -10y + 20x \quad (52b)$$

and $z = 0$. The DAE (52a), (52b) gives $x' = x$ which correctly reflects those of the actual dynamics (50).

Possible advantages of this approach are; retention of dynamic effect of some effects of fast dynamics, being able to use developed DAE integrators, insuring faster contraction in solving the infinite dimensional part by increasing m_2 without increasing number of state variables m_1 that are dynamic, and avoiding some stiffness in the equations being integrated.

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