Nonlinear Observer Design using Implicit System Descriptions

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Abstract

Observers are usually formulated as explicit systems of differential equations and implemented using standard ODE solvers. In this paper, we show that there can be advantages in formulating the observer as a DAE (Differential-Algebraic Equation). We review the general idea of DAE observer design of [10]. We give two special normal forms for which DAE observer design yields an observer with linear error dynamics. The idea to use DAE observer normal forms is introduced on index one DAE observers and then extended to index two Hessenberg DAEs. This allows us to enlarge the class of nonlinear systems for which linear observer error dynamics can be achieved.

1. Introduction

Consider the nonlinear systems

$$\dot{x} = f(x,u) \tag{1}$$
$$y = h(x). \tag{2}$$

where f and h are smooth vector fields on $\mathbb{R}^n$ and $\mathbb{R}^p$, and the p measurements y in (2) are independent. The problem of observer design consist in finding a nonlinear system

$$\begin{aligned} 0 &= \widehat{f}(\dot{\omega},\omega,u,y) \\ \widehat{x} &= \widehat{g}(\omega,u,y) \end{aligned} \tag{3}$$

that generates an estimate $\widehat{x}(t)$ of the true value $x(t)$.

Definition 1 *System (3) is an observer for System (1), if $\widehat{x}(0) = x(0)$ implies $\widehat{x}(t) = x(t)$ for all $t > 0$ and, for $\widehat{x}(0) \neq x(0)$ we have that $\widehat{x}(t) \to x(t)$ as t goes to infinity.*

The problem of how to design system (3) to be an observer for (1) has been extensively studied in the past. There are essentially two approaches. The first approach is a natural extension of linear observers and is very commonly adopted (for example see the techniques presented in the comparative study of [13]). The other approach to observer design is to work directly with system equations (1)-(2), either formulating the estimation problem as a nonlinear algebraic system of equations which must be solved periodically using for example Newton's method (see for example [9]), or formulating it as an optimization problem over some sliding finite horizon which is again solved periodically [8].

In this paper, we present an alternative to these two approaches. We show that there can be advantages in formulating the observer as an implicit (descriptor) system which can then be solved using a DAE (Differential-Algebraic Equation) solver. For index one DAEs this can for example be done by the Differential-Algebraic-System-Solver (DASSL)[3, 11]. More importantly, if (1)-(2) has a special form and verifies some algebraic conditions we can easily construct an DAE index one observer that has linear time invariant observer error dynamics.

The class of nonlinear systems is even larger if we allow (3) to be an index two Hessenberg DAE [3]. Index two Hessenberg DAEs are of particular interest since this type of DAEs can also be safely solved by differential algebraic system solver (for instance with DASSL with fixed stepsize [3]).

The usefulness of this approach will be shown on a simple example. For a more detailed analysis of DAE index two design and its application to mechanical type problems see [12].

2. Index one DAE observer

System (1)-(2) is a DAE in x (u and y are supposed to be known). This over-determined (n unknowns, $n+p$ equations) DAE describes all the constraints (information) that we have for constructing $\hat{x}$. To make this DAE integrable, we can do relaxation by introducing a p-dimensional vector $\lambda(t)$ into this DAE. One way to introduce λ in the algebraic part (the observation) (2)

$$\dot{\hat{x}} = f(\hat{x},u) + g(\lambda) \tag{4}$$
$$0 = y - h(\hat{x}) + \lambda \tag{5}$$

which means that we relax the entire estimate $\widehat{x}$. It easy to see that introducing λ as in (4)-(5) leads to the usual explicit formulation of the observer.

The other way is to introduce λ in the differential part (1). In this case we constrain partially the estimate $\widehat{x}$ through $h(\widehat{x})$ by the observation y and relax only the remaining part. This is the way index one DAE observer design is introduced in [10]. Throughout the remainder of this paper we use the following definition.

Definition 2 *The DAE of the form*

$$\begin{aligned} \dot{\widehat{x}} + \Phi(\widehat{x})\dot{\lambda} &= f(\widehat{x}, u) + \Gamma(\widehat{x}, u)\lambda \quad (6) \\ 0 &= y - h(\widehat{x}) \quad (7) \end{aligned}$$

is called canonical observer DAE for System (1)-(2).

For $\Phi = -h_x(x) = -\partial h(x)/\partial x$ (Γ is any matrix of appropriate dimensions depending on $\widehat{x}$ and u) we recover the canonical index one DAE observer form of [10].

Note that we obtain the original system equations (1)-(2) for $\lambda = 0$. Under appropriate observability conditions, $\lambda(t) \to 0$ implies that $\widehat{x}(t) \to x(t)$ for $\widehat{x}(0) \neq x(0)$. Γ has to be chosen such that $\lambda(t) \to 0$. To this end we observe that for sufficiently small observation error $\widetilde{x} = \widehat{x} - x$ we have

$$\begin{bmatrix} I & -\Phi(\widehat{x}) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\widetilde{x}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} f_x(\widehat{x}) & \Gamma(\widehat{x}, u) \\ h_x(\widehat{x}) & 0 \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ \lambda \end{bmatrix} + O(\|\widetilde{x}\|^2) \quad (8)$$

(where $f_x = \partial f/\partial x$). System (8) has an equilibrium point at $(\widetilde{x}, \lambda) = (0, 0)$. For fixed $(\widehat{x}, u)$ we may chose Γ such that this equilibrium point is stable for all fixed $(\widehat{x}, u)$. If $(\widehat{x}, u)$ varies slowly against $(\lambda, \widetilde{x})$ stability of (8) for fixed $(\widehat{x}, u)$ implies stability for varying $(\widehat{x}, u)$ (see extended linearization [1]). In this case we may use Ackermann's formula (see for example [5]) to compute a $\Gamma(\widehat{x}, u)$.

2.1. Exact linearization of the error equation

If System (1)-(2) is in the special form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u) + F_1(x_1, u)x_2 \\ \dot{x}_2 &= f_2(x_1, u) + F_2(x_1, u)x_2 \quad (9) \\ y &= x_1 \end{aligned}$$

(we will refer to as *index one DAE observer normal form*) and we chose

$$\Phi = \begin{bmatrix} \Omega_1^{-1}(y) \\ \Omega_1^{-1}(y)\Omega_2(y) \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Gamma_1(y) \\ \Gamma_2(y) \end{bmatrix}$$

(6)-(7) is an index one DAE (which will be referred to as *index one canonical DAE observer form*) and (8) becomes the linear time varying system

$$\begin{bmatrix} \dot{\lambda} \\ \dot{\widetilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \Omega_1\Gamma_1 & \Omega_1 F_1 \\ \Gamma_2 - \Omega_2\Gamma_1 & F_2 - \Omega_2 F_1 \end{bmatrix} \begin{bmatrix} \lambda \\ \widetilde{x}_2 \end{bmatrix} \quad (10)$$

There exist a number of methodologies to stabilize (10) by use of Ω and Γ (extended linearization, Lyapunov design, etc.). Here we consider the case where (10) can be made time invariant. For that we need that the matrices F_1 and F_2 have a particular structure.

Theorem 1 *Let $F_2(\widehat{x}_1, u) = \overline{F}_2 + \widetilde{F}_2(\widehat{x}_1, u)$ where $\overline{F}_2$ is a constant matrix. Then, if*

1. *there exists an invertible matrix Ω_1 such that $\Omega_1 F_1$ is constant*
2. *$\overline{F}_2$ can be chosen such that the matrix pair $\{F_1, \overline{F}_2\}$ is observable and there exists a matrix F_2^* such that $\widetilde{F}_2 = F_2^* F_1$*

the error equation (10) can be made time invariant and its modes can arbitrarily be placed by proper choice of Γ_1 and Γ_2 for $\Omega_2 = F_2^*$.

Proof If (13) has Properties 1 and 2, the error equation (19) is

$$\begin{bmatrix} \dot{\lambda} \\ \dot{\widetilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \Omega_1\Gamma_1 & \Omega_1 F_1 \\ \Gamma_2 - F_2^*\Gamma_1 & \overline{F}_2 \end{bmatrix} \begin{bmatrix} \lambda \\ \widetilde{x}_2 \end{bmatrix} \quad (11)$$

Since $\Omega_1 F_1$ is constant and Ω_1 is invertible, (11) is a linear time invariant system for $\Gamma_1 = \Omega_1^{-1} K_1$ and $\Gamma_2 = K_2 + F_2^*\Gamma_1$, where K_1 and K_2 are constant matrices. The modes of (11) can arbitrarily be set by a proper choice of K_1 and K_2 if the matrix pair

$$\left\{ \begin{bmatrix} I & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Omega_1 F_1 \\ 0 & \overline{F}_2 \end{bmatrix} \right\}$$

is observable which is the case if the matrix pair $\{\Omega_1 F_1, \overline{F}_2\}$ is observable. □

3. Index two DAE observer

The idea to use DAEs as observer for (1)-(2) can be extended to the case where (6)-(7) is in Hessenberg semi explicit index two form.

Definition 3 *A DAE is in Hessenberg semi explicit index two[1] form if it has the structure*

$$\begin{aligned} \dot{\omega}_1 &= \widehat{f}(t, \omega_1, \omega_2) \\ 0 &= \widehat{g}(t, \omega_1) \end{aligned} \quad (12)$$

where $(\partial\widehat{g}/\partial\omega_1)(\partial\widehat{f}/\partial\omega_2)$ is nonsingular.

We use the following result from [3], Theorem 3.2.2 :

Lemma 1 *Suppose the nonlinear semi-explicit index two system (12), is to be solved numerically be the k-step BDF [2] method ($k < 7$) with fixed step size h, the errors in the initial values are $\|e_0\| = O(h^k)$, and the errors in terminating the Newton iteration satisfy $O(h^{k+1})$. Then the k-step BDF method is convergent and globally accurate to $O(h^k)$, after $k+1$ steps.*

The generalization of DAE index one observe design is for example of interest in the case where (1)-(2) can be put in the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_2(x_1, x_2, u) + F_2(x_1, u)x_3 \\ \dot{x}_3 &= f_3(x_1, x_2, u) + F_3(x_1, u)x_3 \\ y &= x_1 \end{aligned} \quad (13)$$

[1] It can easily be verified that two differentiations of the algebraic equation $0 = \widehat{g}(\omega_1)$ suffice to transform (12) into an ODE.

[2] Backward Differential Formula, see for example [4].

This particular form is frequently encountered in mechanical type systems (flexible joint robots are for example in this form [12]) and will be refered to as *index two DAE observer normal form.*The observer is the canonical observer DAE (6)-(7) with

$$\Phi = \begin{bmatrix} 0 \\ \Omega_1^{-1}(y) \\ \Omega_1^{-1}(y)\Omega_2(y) \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 0 \\ \Gamma_1(y,\hat{x}_2,u) \\ \Gamma_2(y,\hat{x}_2,u) \end{bmatrix}$$

which we will refer to as *index two canonical DAE observer form.* To insure integrability, we need to show that the index two canonical DAE observer form can be transformed into Hessenberg form.

Lemma 2 *The change of coordinates*

$$\begin{array}{rcl} \omega_1 &=& \hat{x}_1 \\ \omega_2 &=& \hat{x}_2 \\ \omega_3 &=& \hat{x}_3 - \Omega_2(\hat{x}_1)\hat{x}_2 \\ \omega_4 &=& \lambda + \Omega_1(\hat{x}_1)\hat{x}_2 \end{array} \tag{14}$$

transforms (13) into the Hessenberg semi explicit index two form

$$\begin{array}{rcl} \dot{\omega}_1 &=& \omega_2 \\ \dot{\omega}_3 &=& \hat{f}_2(\omega_1,\omega_2,\omega_4,u) + \hat{F}_2(\omega_1,\omega_2,u)\omega_3 \\ \dot{\omega}_4 &=& \hat{f}_3(\omega_1,\omega_2,\omega_4,u) + \hat{F}_3(\omega_1,\omega_2,u)\omega_3 \\ 0 &=& y - \omega_1 \end{array} \tag{15}$$

where

$$\begin{array}{l} \hat{f}_2 = f_3 - \Omega_2 f_2 - \left(\dot{\Omega}_2 - F_3\Omega_2 + \Omega_2 F_2 \Omega_2\right)\omega_2 + \\ \quad (\Gamma_2 - \Omega_2\Gamma_1)\lambda \\ \hat{f}_3 = \Omega_1 f_2 + \Omega_1\left(\dot{\Omega}_1 + F_2\Omega_2\right)\omega_2 + \Omega_1\Gamma_1\lambda \\ \hat{F}_2 = F_3 - \Omega_2 F_2, \qquad \hat{F}_3 = \Omega_1 F_2 \end{array} \tag{16}$$

and $\lambda = \omega_4 - \Omega_1\omega_1$.

Proof We have

$$\dot{\omega}_3 = \dot{\hat{x}}_3 - \dot{\Omega}_2\hat{x}_2 - \Omega_2\dot{\hat{x}}_2 \tag{17}$$

$$\dot{\omega}_4 = \dot{\lambda} + \dot{\Omega}_1\hat{x}_2 + \Omega_1\dot{\hat{x}}_2 \tag{18}$$

Elimination of $\Omega_2\dot{\hat{x}}_2$ in (17) by use of (18) and the left multiplication of (18) by Ω_1^{-1} yields

$$\dot{\omega}_3 + \Omega_2\Omega_1^{-1}\dot{\omega}_4 = \left\{\dot{\hat{x}}_3 + \Omega_2\Omega_1^{-1}\dot{\lambda}\right\} - \left(\dot{\Omega}_2 - \Omega_2\Omega_1^{-1}\dot{\Omega}_1\right)\hat{x}_2$$

$$\Omega_1^{-1}\dot{\omega}_4 = \left\{\dot{\hat{x}}_2 + \Omega_1^{-1}\dot{\lambda}\right\} + \Omega_1^{-1}\dot{\Omega}_1\hat{x}_2$$

The sums in the curly brackets at the right hand side are the left hand side of the of the third and second block row of the index two DAE observer form. Together with (14) and (15) we get

$$\begin{array}{l} 0 = \hat{f}_2 + \Omega_2\Omega_1^{-1}\hat{f}_3 + \left(\hat{F}_2 + \Omega_2\Omega_1^{-1}\hat{F}_3\right)(\hat{x}_3 - \Omega_2\hat{x}_2) - \\ \quad f_3 - F_3\hat{x}_3 - \Gamma_2\lambda + \left(\dot{\Omega}_2 - \Omega_2\Omega_1^{-1}\dot{\Omega}_1\right)\hat{x}_2 \\ 0 = \Omega_1^{-1}\hat{f}_3 + \Omega_1^{-1}\hat{F}_3(\hat{x}_3 - \Omega_2\hat{x}_2) - f_2 - F_2\hat{x}_3 - \Gamma_1\lambda - \\ \quad \Omega_1^{-1}\dot{\Omega}_1\hat{x}_2 \end{array}$$

which yield the definitions (16). □

3.1. Exact linearization of the error equation

We proceed like in the index one case: due to the assumption that (1) is in index two DAE observer normal form (13), the error equation is the linear time varying system.

$$\begin{bmatrix} \dot{\lambda} \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} \Omega_1\Gamma_1 & \Omega_1 F_2 \\ \Gamma_2 - \Omega_2\Gamma_1 & F_3 - \Omega_2 F_2 \end{bmatrix} \begin{bmatrix} \lambda \\ \tilde{x}_3 \end{bmatrix} \tag{19}$$

To get a linear error equation we need that F_2 and F_3 have a particular structure.

Theorem 2 *Let* $F_3(\hat{x}_1,\hat{x}_2,u) = \overline{F}_3 + \tilde{F}_3(\hat{x}_1,\hat{x}_2,u)$ *where* $\overline{F}_3$ *is a constant matrix. Then, if*

1. *there exists an invertible matrix* Ω_1 *such that* $\Omega_1 F_2$ *is constant*
2. $\overline{F}_3$ *can be chosen such that the matrix pair* $\{F_2, \overline{F}_3\}$ *is observable and there exists a matrix* F_3^* *such that* $\tilde{F}_3 = F_3^* F_2$,

the error equation (19) can be made time invariant and its modes can arbitrarily be placed by proper choice of Γ_1 *and* Γ_2 *for* $\Omega_2 = F_3^*$.

Proof The proof is similar to that of Theorem 1. □

4. Example

We use as example the model of a three-phase current motor, which is also used in [2] and [10]. We show on this model how to apply index one DAE observer design if (1)-(2) is in index one DAE normal form. Furthermore, we show that DAE index two observer design may yield linear error dynamics even if DAE index one observer design does not. This shows that the extension of DAE observer design to index two DAEs allows us to enlarge the class of nonlinear systems with linearizable error dynamics.
We consider the following system.

$$\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = B_1 - A_1 x_2 - A_2 x_3 \sin(x_1) + \frac{1}{2}\sin(2x_1) \\ \dot{x}_3 = u - D_1 x_3 + D_2\cos(x_1) \end{array} \tag{20}$$

where $x = (x_1, x_2, x_3)^T$ is the state, u a control input and B_1, A_1, A_2 , D_1 and D_2 are constants.

4.1. Index one observer

For the index one DAE observer design we use the output

$$y = (x_1, x_2)^T \tag{21}$$

It can easily be seen that System (20) is in index one DAE normal form (9). We have

$$f_1(y,u) = \begin{bmatrix} x_2 \\ B_1 - A_1 x_2 + \frac{1}{2}\sin(2x_1) \end{bmatrix}$$

$$F_1(y,u) = \begin{bmatrix} 0 \\ -A_2\sin(x_1) \end{bmatrix}$$

$$f_2(y,u) = u + D_2\cos(x_1), \quad F_2(y,u) = -D_1$$

To see that (20) has for the output (21) a linear time invariant error equation we need the show that F_1 and F_2 verify Theorem 1. As $F_2(y) = \overline{F}_2$ and for

$$\Omega_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{k_3}{A_2 \sin(x_1)} \end{bmatrix} \quad (22)$$

(k_3 is a constant scalar) the product $\Omega_1 F_1 = (0, k_3)^T$ is a constant matrix, the observer error admits a linear time invariant error equation. To see that the poles of the error equation can be placed at arbitrary locations we need to show that the matrix pair $\{F_1, \overline{F}_2\}$ is observable, that is, we need to show that

$$\text{rank} \begin{bmatrix} \overline{F}_2 \\ F_1 \overline{F}_2 \\ \vdots \end{bmatrix} = 1$$

which is the case since $\overline{F}_2 = -D_1$ is a constant scalar. The result is consistent with [2] where it is shown that System (20) admits a linear time invariant error equation for the output (21). For

$$\Gamma_1 = \begin{bmatrix} k_1 & 0 \\ 0 & -\frac{k_2}{k_3} A_2 \sin(x_1) \end{bmatrix}, \ \Gamma_2 = (k_4, k_5), \ \Omega_2 = (0, 0)$$

and (22) we obtain the DAE index one observer

$$\begin{aligned} \dot{\widehat{x}}_1 &= \widehat{x}_2 - \dot{\lambda}_1 + k_1 \lambda_1 \\ \dot{\widehat{x}}_2 &= B_1 - A_1 \widehat{x}_2 - A_2 \widehat{x}_3 \sin(\widehat{x}_1) + \tfrac{1}{2} \sin(2\widehat{x}_1) + \\ &\quad A_2 \sin(\widehat{x}_1) \tfrac{1}{k_3} (\dot{\lambda}_2 - k_2 \lambda_2) \\ \dot{\widehat{x}}_3 &= u - D_1 \widehat{x}_3 + D_2 \cos(\widehat{x}_1) + k_4 \lambda_1 + k_5 \lambda_2 \\ y_1 &= \widehat{x}_1 \\ y_2 &= \widehat{x}_2 \end{aligned}$$

which has the linear time invariant error equation

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & k_3 \\ k_4 & k_5 & -D_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \widetilde{x}_3 \end{bmatrix}$$

Now, instead of two observations, we take just one

$$y = x_1 \quad (23)$$

It can easily be seen that for this observation System (20) is still in index one DAE observer normal form (9) but Theorem 1 does not hold any longer. More importantly, the conditions of nonlinear observer design of [2] or [7] (or [6] for the single output case) applied on System (20) show that System (20) cannot be transformed into nonlinear observer form and, consequently, that it has no linear error equation for the output (23). [6] gives sufficient conditions for the single output case. Let System (20) define $\dot{x} = f(x)$, where $x = (x_1, x_2, x_3)^T$ and $y = h(x)$ be the system output. By Proposition 3 of [6] the auxiliary vector field $g(x)$ defined by

$$g(x) = L_g L_f^k(h) = \begin{cases} 0, & 0 \le k \le n-1 \\ 1, & k = n-1 \end{cases}$$

is $g(x)^T = (0, 0, -(A_2 \sin(x_1))^{-1})^T$. To admit a linear error equation the brackets $[g, \text{ad}^1(f)g]$ and $[g, \text{ad}^3(f)g]$ must be zero. We obtain that $[g, \text{ad}^1(f)g] = 0$, but $[g, \text{ad}^3(f)g] \neq 0$; hence, system (20) admits no linear error equation.

4.2. Index two observer

If we use DAE index two observer design for System (20) with the output (23) we obtain a linear time invariant observer error equation.

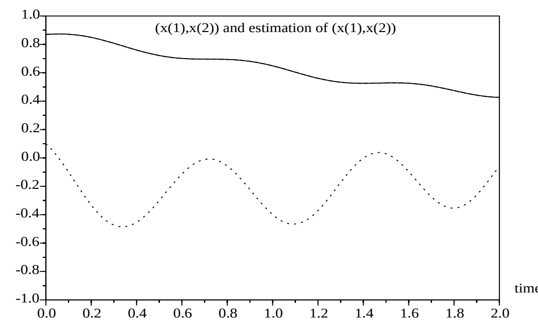

Figure 1 *Without perturbation the estimate $\widehat{x}_1$ and x_1 (solid line), and $\widehat{x}_2$and x_2 (dotted line) are identical for all $t > 0$. However, $\widehat{x}_1$ may have a jump at $t = 0$ if $\widehat{x}_1(0) \neq y(0)$. $\widehat{x}_2$ is impulsive at $t = 0$ if $\widehat{x}_1(0) \neq y(0)$ and has a jump if $\widehat{x}_2(0) \neq \dot{y}(0)$.*

In fact, System (20) is in index two DAE observer normal form (14), where

$$\begin{aligned} f_2(y, \dot{y}, u) &= B_1 - A_1 x_2 + \tfrac{1}{2} \sin(2x_1) \\ F_2(y, \dot{y}, u) &= -A_2 \sin(x_1) \\ f_3(y, \dot{y}, u) &= D_2 \sin(x_1) + u \\ F_3(y, \dot{y}, u) &= -D_3 \end{aligned}$$

Furthermore, F_3 and F_2 verify Theorem 2 since we have $F_3 = \overline{F}_3$, the choice

$$\Omega_1 = -\kappa_2 / (A_2 \sin(x_1)) \quad (24)$$

yields $\Omega_1 F_2 = \kappa_2$ and the pair $\{F_2, \overline{F}_3\}$ is observable as $\overline{F}_3 = D_1 = \text{const.}$. For $\Gamma_1 = -\kappa_1 A_2 \sin(x_1)/\kappa_2$, $\Gamma_2 = \kappa_3$ and $\Omega_2 = 0$ the DAE index two observer is

$$\begin{aligned} \dot{\widehat{x}}_1 &= \widehat{x}_2 \\ \dot{\widehat{x}}_2 &= B_1 - A_1 \widehat{x}_2 - A_2 \widehat{x}_3 \sin(\widehat{x}_1) + \tfrac{1}{2} \sin(2\widehat{x}_1) + \\ &\quad A_2 \sin(x_2) \tfrac{1}{\kappa_2} (\dot{\lambda} - \kappa_1 \lambda) \\ \dot{\widehat{x}}_3 &= u - D_1 \widehat{x}_3 + D_2 \cos(\widehat{x}_1) + \kappa_3 \lambda \\ y &= \widehat{x}_1 \end{aligned}$$

which is an index two DAE provide $\widehat{x}_1 \neq \pi k, \ k = 0, \pm 1, \pm 2, \cdots$. The error equation is

$$\begin{bmatrix} \dot{\lambda} \\ \dot{\tilde{x}}_3 \end{bmatrix} = \begin{bmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & -D_1 \end{bmatrix}$$

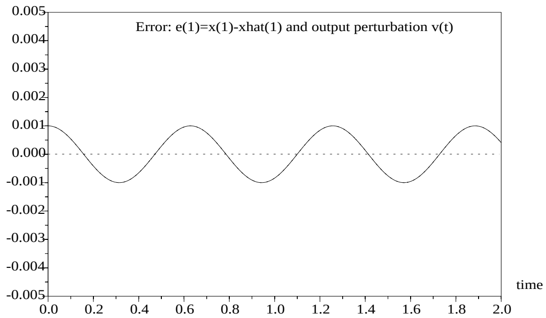

Figure 2 *The estimation error for x_1 is zero for all $t > 0$ (dotted line). With perturbation (solid line) the estimation error is of the size of the perturbation v.*

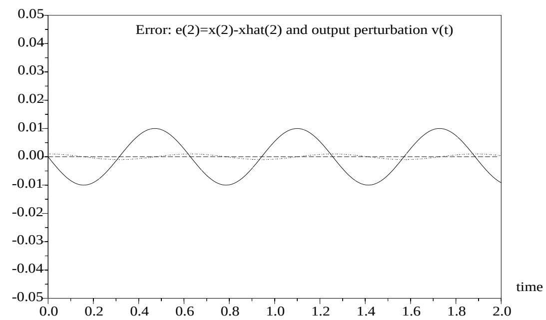

Figure 3 *The estimation error for x_2 is zero for all $t > 0$ (dashed line). With perturbation (solid line) the estimation error is the first derivative of the perturabation v (dotted line).*

We note that the index two DAE observer for System (20) is obtained from the index one DAE observer if we eliminate $\dot{\lambda}_1$ in the first line, set $\lambda_2 = \lambda$, $k_1 = 0$, $k_2 = \kappa_1$, $k_3 = \kappa_2$ $k_4 = 0$ and $k_5 = \kappa_3$.
To insure integrability we need to transform the index two DAE into Hessenberg form, that is we need that the observer (3) be in the special form (15). We have for the observer (3)

$$0 = \begin{bmatrix} \dot{\omega}_1 - \omega_2 \\ \dot{\omega}_3 - \widehat{f}_2(\omega_1, \omega_2, \omega_4, u) - \widehat{F}_2(\omega_1, \omega_2, u)\omega_3 \\ \dot{\omega}_4 - \widehat{f}_3(\omega_1, \omega_2, \omega_4, u) - \widehat{F}_3(\omega_1, \omega_2, u)\omega_3 \\ y - \omega_1 \end{bmatrix}$$
$$(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = (\omega_1, \omega_2, \omega_3)$$

where $\widehat{f}_2 = D_2 \sin(\omega_1) + u + \kappa_3 \lambda$,

$$\widehat{f}_3 = \frac{-\kappa_2}{A_2 \sin(\omega_1)} \left(B_1 - A_1 \omega_2 + \frac{1}{2} \sin(2\omega_1) + \frac{\kappa \omega_2^2 \cos \omega_1}{A_2 \sin^2(\omega_1)} \right) + \kappa_1 \lambda$$

$\widehat{F}_2 = -A_2 \sin(\omega_1)$, $\widehat{F}_3 = \kappa_2$ and $\lambda = \omega_4 + \kappa_2 \omega_1 / (A_2 \sin(\omega_1)$. The resulting DAE is in Hessenberg semi explicit index two form, which can safely be integrated by BDF integration schemes with fixed stepsize.

4.2.1 Numeric Simulation: For the following computer simulation we use the same values for the model parameters as in [10] and [2]:

A_1	A_2	B_1	B_2	D_1	D_2
0.2703	12.01	39.19	-48.04	0.3222	1.9

The index two DAE observer includes a sort of numeric differentiation. To show the impact of a perturbation in the observation we have perturbed the constraint by $v(t) = 0.001 \cos(10t)$, i.e., the perturbed constraint is $0 = \hat{x}_1 - y + v$. The perturbation on $\widehat{x}_1$ is obviously $v(t)$ and that on $\widehat{x}_2$ is $\dot{v}(t) = -0.01 \cos(10t)$. If the observation is not perturbed $\widehat{x}_2$ jumps immediately to its true value x_2. If $\widehat{x}_1(0) \neq y(0)$ the observation $\widehat{x}_2$ is impulsive at $t = 0$. The simulation

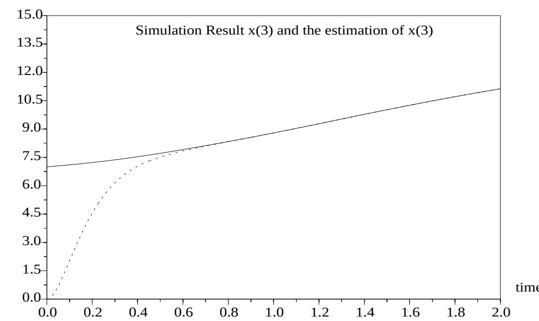

Figure 4 *Independently of the perturbation the estimate $\widehat{x}_3$ (dotted line) converges to the true value x_3 (solid line). If the observation is perturbed $\widehat{x}_3$ remains in the neighborhood of x_3.*

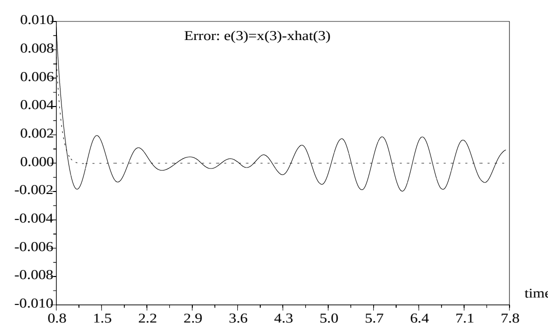

Figure 5 *Estimation error on x_3 with (solid line) and without perturbation (dotted line).*

shows that the additional state λ in the observer is a relaxation for hidden constraints. The constraint

$$0 = y - \widehat{x}_1 \tag{25}$$

includes two hidden constraints:

$$0 = \dot{y} - \widehat{x}_2 \tag{26}$$
$$0 = \ddot{y} - \widehat{f}_2(\widehat{x}_1, \widehat{x}_2, \lambda) \tag{27}$$

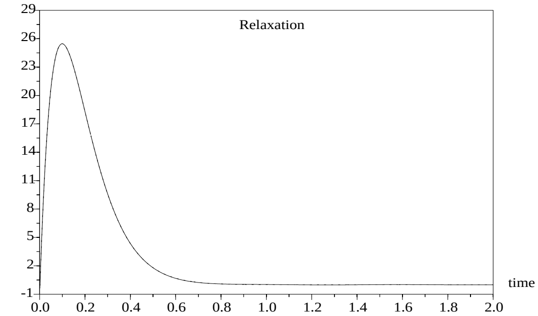

Figure 6 *Additional state λ.*

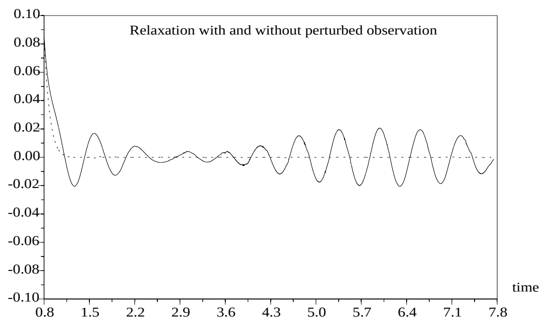

Figure 7 *Without perturbation $\lambda(t)$ goes to zero for growing time. If the observation is perturbed λ stays in the neighborhood of zero. λ is a measurement of the violation of the hidden constraint (27) and the observation error $\widetilde{x}_3$.*

where $\widehat{f}_2(\widehat{x}_1, \widehat{x}_2, \lambda) = (B_1 - A_1\hat{x}_2 - A_2\hat{x}_3\sin(\hat{x}_1) + \frac{1}{2}\sin(2\hat{x}_1) + \Omega_1^{-1}\lambda + \Gamma_1\lambda)$. The integration by BDF integration schemes assures that the hidden constraint (26) is verified for all $t > 0$. The hidden constraint (27) is relaxed by the supplementary state λ. The constraint error $\ddot{y} - \widehat{f}_2$ goes asymptoticly to zero if $\lambda \to 0$.

5. Conclusion

We have shown that by a slight generalization of index one DAE observer design of [10] and the introduction of a normal form, DAE index one observer design yields, applied on a special class of nonlinear systems (which is defined by the normal form) observers with linear error dynamics. We have given easy to test conditions for which the resulting error equation can be made, in addition, time invariant. If these conditions are verified the construction of an index one DAE observer with linear time invariant error dynamics is straightforward. We have shown that the idea of DAE observer design can be extended to the case where the DAE observer is in Hessenberg semi explicit index two form. We have given a second normal form for which index two DAE observer design yields linear error dynamics. The conditions for which we have a linear time invariant error equation are similar to that of the index one case. We have shown on a simple example that index two DAE observer design can yield a linear error equation even if index one DAE observer design fails.

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