OBSERVER DESIGN FOR LINEAR TIME VARYING DESCRIPTOR SYSTEMS

S. L. Campbell [*,1] **F. Delebecque** [**] **R. Nikoukhah** [***]

[*] *Department of Mathematics, North Carolina State University, Raleigh, NC, USA 27695-8205, email: slc@math.ncsu.edu*
[**] *INRIA, Rocquencourt BP 105, 78513 Le Chesnay Cedex, France*
[***] *INRIA, Rocquencourt BP 105, 78513 Le Chesnay Cedex, France*

Abstract: This paper examines the design of observers for linear time varying descriptor systems. An observer is designed for which the observer estimates are also physically correct in that they satisfy the same constraints as the solutions of the descriptor system. Careful attention is paid to ensuring that all quantities are computable by numerical algorithms.

Résumé: On étudie dans ce papier la conception d'observateurs pour des systèmes linéaires à temps variant. On définit un observateur pour lequel les variables estimées sont physiquement admissibles au sens où elles satisfont les mêmes contraintes que les solutions du système descripteur. Une attention toute particulière est portée aux algorithmes numériques nécessaires au calcul de toutes les quantités utilisées.

Keywords: Descriptor systems, Observers, Linear Systems, Time-varying systems, Computational methods.

Mots clés: Systèmes descripteurs, Observateurs, Systèmes linéaires, Systèmes à temps variant, Techniques numériques.

1. INTRODUCTION

Many physical systems are most easily initially modeled as systems of differential and algebraic equations (Brenan *et al.*, 1996). These systems are variously called descriptor systems, singular systems, or differential algebraic equations (DAEs). Reduction to an explicit model may require simplification with the accompanying loss of accuracy, loss of sparsity, substantial effort, or even be impossible.

There has been a considerable of amount of research done on the design of observers for linear time invariant descriptor systems (Dai, 1989; Hou *et al.*, 1993; Pearson *et al.*, 1988; Shin and Kabamba, 1988). However, many descriptor systems of interest are nonlinear or linear time varying. The use of time invariant linearizations sometimes results in incorrect approximations (Campbell, 1995). Nonlinear systems, if correctly linearized along a trajectory, naturally result in linear time varying DAEs. There has been very little work on the design of observers for general linear time varying DAEs.

This paper will examine the design of observers for general linear time varying DAEs. Careful attention will be paid to which quantities are known and which are computed to make sure that the design can actually be implemented. Much of the existing literature on nonlinear or time varying systems makes extensive use of nonlinear coordinate changes and differentiations of computed

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quantities. While this is theoretically attractive, it sometimes requires symbolic manipulation in order to actually implement the approach. This is often not practical for large or complex systems due to expression swell. The approach in this paper is different. In this paper, a computable quantity is one that is derived only by differentiations of the *original* equations followed by numerical calculations. The differentiation need not be done symbolically. Often the best option will be to use automatic differentiation (AD) codes (Campbell and Hollenbeck, 1996). AD software rapidly computes first and higher order derivatives at a given time up to round off error without the expression swell and memory use of a symbolic approach. Thus, while differentiation may be permitted in a proof or the verification of a formula, no explicit differentiation of computed quantities is permitted in the algorithms. How they can be reliably computed in cases where derivatives are needed will be discussed. It is assumed that the reader is familiar with basic DAE properties such as the index (Brenan *et al.*, 1996). In order to keep track of how the various information is derived, a hat will often be used for computed quantities.

2. THE BASIC PROBLEM

The basic system to be considered is the descriptor system

$$\mathbf{E}(t)\mathbf{x}' = \mathbf{F}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} + \mathbf{g}(t) \tag{1}$$

$$\mathbf{y} = \mathbf{H}(t)\mathbf{x} \tag{2}$$

where $\mathbf{u}$ is a known control. $\mathbf{g}$ occurs because the system may have come from linearization along a function which may or may not be a solution of the original system. It is assumed that $\mathbf{E}, \mathbf{F}, \mathbf{B}, \mathbf{H}, \mathbf{g}$ are known matrix valued functions. $\mathbf{E}$ is not assumed to have constant rank. The observer design problem considered here is to find $\mathbf{K}, \mathbf{L}, \mathbf{M}, \mathbf{R}$ so that the system

$$\mathbf{K}(t)\mathbf{z}' = \mathbf{L}(t)\mathbf{z} + \mathbf{M}(t)(\mathbf{y} - \mathbf{H}(t)\mathbf{z}) \tag{3}$$

$$\mathbf{w} = \mathbf{R}(t)\mathbf{z} \tag{4}$$

has the properties:

(P1) $\lim_{t\to\infty} \|\mathbf{x} - \mathbf{w}\| = 0$.

(P2) System (3)-(4) can be numerically integrated when the observer is implemented.

(P3) The degree of freedom in $\mathbf{w}$ (dimension of the solution manifold of (3) projected onto $\mathbf{w}$ by $\mathbf{R}$) is the same as the dimension of the solution manifold of (1).

Requirement (P3) is motivated by problems where the constraint manifold represents physical constraints. The goal is to have an observation which is also physically correct. Note that because $\mathbf{K}(t)$ is singular, (4) includes the possibility that $\mathbf{w}$ depends directly on $\mathbf{y}, \mathbf{z}$.

3. IMPLICIT CONSTRAINT PRESERVING APPROACH

In this paper, the only nonnumerical operation allowed in algorithms is possibly the differentiation of the known functions $\mathbf{E}, \mathbf{F}, \mathbf{B}, \mathbf{H}, \mathbf{u}, \mathbf{q}$. A direct duplication of the usual theory would take $\mathbf{K} = \mathbf{E}$ and $\mathbf{L} = \mathbf{F}$ in (3). However, $\mathbf{z}$ would then involve derivatives of $\mathbf{y}$ if $\mathbf{K}\mathbf{z}' + \mathbf{L}\mathbf{z}$ is a higher index DAE. In this paper $\mathbf{u}$ is considered known so that while not desirable, implicit or explicit differentiation of $\mathbf{u}$ will be allowed. However, $\mathbf{y}$ is usually measured. Differentiation of $\mathbf{y}$ is to be avoided. Accordingly, a different approach must be considered.

Section 6 gives algorithms for numerically computing $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{g}}, \widehat{\mathbf{G}}, \widehat{\mathbf{h}}, \widehat{\mathbf{v}}$ such that the solutions of (1)-(2) are precisely the solutions of

$$\mathbf{x}' = \widehat{\mathbf{A}}\mathbf{x} + \widehat{\mathbf{B}}\mathbf{v} + \widehat{\mathbf{g}} \tag{5}$$

$$\mathbf{0} = \widehat{\mathbf{G}}(t)\mathbf{x} + \widehat{\mathbf{h}} \tag{6}$$

$$\mathbf{y} = \mathbf{H}(t)\mathbf{x} \tag{7}$$

$\mathbf{v}$ is composed of derivatives of the known $\mathbf{u}$. $\widehat{\mathbf{h}}$ depends on $\mathbf{v}, \mathbf{g}$. (6) is an invariant of (5) for this $\mathbf{u}$ and characterizes the consistent initial conditions (solution manifold) at time t. (5)-(6) will be referred to as the *explicit representation* of (1) and (5)-(7) as the explicit representation of (1)-(2). (6) is needed in order to insure (P3). However, as shown in Example 1, a system that is stabilizable with (6) might not be stabilizable without (6).

It is important to note that $\widehat{\mathbf{A}}$ is not unique and depends on the way the completion is computed. In general it is possible to compute any matrix coefficient of the form $\widehat{\mathbf{A}} + \widetilde{\mathbf{\Theta}}\widehat{\mathbf{G}}$ (Campbell, 1992). If the numerical algorithm computing the coefficients makes discrete decisions, such as choice of pivots, $\widetilde{\mathbf{\Theta}}$ can depend discontinuously on t. $\widetilde{\mathbf{\Theta}}$ will be continuous if a numerical method like the singular value decomposition (SVD) is used. However, all possible $\widehat{\mathbf{A}}$ agree on the nullspace of $\widehat{\mathbf{G}}$. There is uniqueness of $\widehat{\mathbf{A}}$ if (6) is enforced but not if (6) not enforced since all completions agree on the solution manifold. This is another reason to want the observer to satisfy these same constraints.

An obvious first form for an observer of (5)-(7), if (6) is preserved, would be

$$\widehat{\mathbf{x}}' = \widehat{\mathbf{A}}\widehat{\mathbf{x}} + \widehat{\mathbf{B}}\mathbf{v} + \widehat{\mathbf{g}} + \mathbf{L}(\mathbf{y} - \mathbf{H}\widehat{\mathbf{x}}) \tag{8}$$

$$\mathbf{0} = \widehat{\mathbf{G}}(t)\widehat{\mathbf{x}} + \widehat{\mathbf{h}} \tag{9}$$

where $\widehat{\mathbf{L}}$ is to be determined. There are two different issues. One is to have an algorithm for computing $\widehat{\mathbf{L}}$. The second problem is that given that $\widehat{\mathbf{L}}$ is determined, it is still necessary to be able to integrate the DAE (8)-(9).

3.1 *The feedback* $\widehat{\mathbf{L}}$

Let $\boldsymbol{\varepsilon} = \widehat{\mathbf{x}} - \mathbf{x}$ be the error in the observer (8)-(9). Then (5)-(9) imply that $\boldsymbol{\varepsilon}$ satisfies the error equation

$$\boldsymbol{\varepsilon}' = \widehat{\mathbf{A}}\boldsymbol{\varepsilon} - \widehat{\mathbf{L}}\mathbf{H}\boldsymbol{\varepsilon} \tag{10}$$
$$\mathbf{0} = \widehat{\mathbf{G}}\boldsymbol{\varepsilon} \tag{11}$$

The nullspace of $\widehat{\mathbf{G}}$ is computable and independent of the particular algorithm used to get $\widehat{\mathbf{G}}$. Let $\widehat{\mathbf{P}} = \mathbf{I} - \widehat{\mathbf{G}}^{\dagger}\widehat{\mathbf{G}}$ which is the unique orthogonal projection onto the nullspace of $\widehat{\mathbf{G}}$ (Campbell and Meyer, 1991). This will be smooth since the nullspace of $\widehat{\mathbf{G}}$ varies smoothly. If $\widehat{\mathbf{G}}'$ is known, then $\widehat{\mathbf{P}}'$ is computable. How to do this will be discussed in Section 6.

Let a smooth SVD of $\widehat{\mathbf{P}}$ be

$$\widehat{\mathbf{P}} = \mathbf{U}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^T = \mathbf{U}\widetilde{\mathbf{P}}\mathbf{U}^T \tag{12}$$

Note that $\widetilde{\mathbf{P}}$ is constant. The error equation (10)-(11) can be written

$$\boldsymbol{\varepsilon}' = (\widehat{\mathbf{A}} - \widehat{\mathbf{L}}\mathbf{H})\boldsymbol{\varepsilon} \tag{13}$$
$$\mathbf{0} = (\mathbf{I} - \widehat{\mathbf{P}})\boldsymbol{\varepsilon} \tag{14}$$

or equivalently,

$$(\widehat{\mathbf{P}}\boldsymbol{\varepsilon})' = (\widehat{\mathbf{A}} - \widehat{\mathbf{L}}\mathbf{H})\widehat{\mathbf{P}}\boldsymbol{\varepsilon} \tag{15}$$
$$\mathbf{0} = (\mathbf{I} - \widehat{\mathbf{P}})\boldsymbol{\varepsilon} \tag{16}$$

Let $\boldsymbol{\eta} = \mathbf{U}^T\boldsymbol{\varepsilon}$. Then this system is

$$(\mathbf{U}\widetilde{\mathbf{P}}\boldsymbol{\eta})' = (\widehat{\mathbf{A}} - \widehat{\mathbf{L}}\mathbf{H})\mathbf{U}\widetilde{\mathbf{P}}\boldsymbol{\eta} \tag{17}$$
$$\mathbf{0} = \mathbf{U}(\mathbf{I} - \widetilde{\mathbf{P}})\boldsymbol{\eta} \tag{18}$$

Thus

$$\mathbf{U}'\widetilde{\mathbf{P}}\boldsymbol{\eta} + \mathbf{U}\widetilde{\mathbf{P}}\boldsymbol{\eta}' = (\widehat{\mathbf{A}} - \widehat{\mathbf{L}}\mathbf{H})\mathbf{U}\widetilde{\mathbf{P}}\boldsymbol{\eta} \tag{19}$$
$$\mathbf{0} = \boldsymbol{\eta}_2 \tag{20}$$

Define

$$-\mathbf{U}^T\mathbf{U}' = \begin{bmatrix}\boldsymbol{\Theta}_1 & \boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_3 & \boldsymbol{\Theta}_4\end{bmatrix}, \mathbf{U}^T\mathbf{A}\mathbf{U} = \begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix} \tag{21}$$

where the partition is conformal with $\widetilde{\mathbf{P}}$. Multiplying (19) on the left by $\mathbf{U}^T$ and combining terms gives

$$\boldsymbol{\eta}_1' = (\boldsymbol{\Theta}_1 + \mathbf{A}_1)\boldsymbol{\eta}_1 - (\mathbf{U}^T\widehat{\mathbf{L}}\mathbf{H}\mathbf{U})_{11}\boldsymbol{\eta}_1 \tag{22}$$
$$\mathbf{0} = (\boldsymbol{\Theta}_3 + \mathbf{A}_3)\boldsymbol{\eta}_1 - (\mathbf{U}^T\widehat{\mathbf{L}}\mathbf{H}\mathbf{U})_{21}\boldsymbol{\eta}_1 \tag{23}$$
$$\mathbf{0} = \boldsymbol{\eta}_2 \tag{24}$$

The invariance of the range of $\widehat{\mathbf{P}}$ implies that $\boldsymbol{\Theta}_3 + \mathbf{A}_3 = \mathbf{0}$. On the other hand, the error equations are to have the same dimensional solution manifold as $\mathbf{x}, \widehat{\mathbf{x}}$. Thus (23) must hold without constraining $\boldsymbol{\eta}_1$. This gives

$$\boldsymbol{\eta}_1' = (\boldsymbol{\Theta}_1 + \mathbf{A}_1)\boldsymbol{\eta}_1 - (\mathbf{U}^T\widehat{\mathbf{L}}\mathbf{H}\mathbf{U})_{11}\boldsymbol{\eta}_1 \tag{25}$$
$$\mathbf{0} = \boldsymbol{\eta}_2 \tag{26}$$

with the requirement that $\widehat{\mathbf{L}}$ be chosen so that

$$(\mathbf{U}^T\widehat{\mathbf{L}}\mathbf{H}\mathbf{U})_{21} = \mathbf{0} \tag{27}$$

Partition the orthogonal matrix $\mathbf{U}$ as

$$\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2] \tag{28}$$

Then (25)-(26) is

$$\boldsymbol{\eta}_1' = (\boldsymbol{\Theta}_1 + \mathbf{A}_1)\boldsymbol{\eta}_1 - (\mathbf{U}_1^T\widehat{\mathbf{L}}\mathbf{H}\mathbf{U}_1)\boldsymbol{\eta}_1 \tag{29}$$
$$\mathbf{0} = \boldsymbol{\eta}_2 \tag{30}$$

with the requirement (27) that

$$\mathbf{U}_2^T\widehat{\mathbf{L}}\mathbf{H}\mathbf{U}_1 = \mathbf{0} \tag{31}$$

Let $\widehat{\mathbf{L}} = \mathbf{U_1}\overline{\mathbf{K}}$. Then (31) holds and (29)-(30) is

$$\boldsymbol{\eta}_1' = (\boldsymbol{\Theta}_1 + \mathbf{A}_1)\boldsymbol{\eta}_1 - (\overline{\mathbf{K}}\mathbf{H}\mathbf{U}_1)\boldsymbol{\eta}_1 \tag{32}$$
$$\mathbf{0} = \boldsymbol{\eta}_2 \tag{33}$$

$\overline{\mathbf{K}}$ has to be chosen so that (32) is stable. That is, it must solve the output stabilization problem

$$\boldsymbol{\eta}_1' = (\boldsymbol{\Theta}_1 + \mathbf{A}_1)\boldsymbol{\eta}_1 + \mathbf{u} \tag{34}$$
$$\mathbf{y} = \mathbf{H}\mathbf{U}_1\boldsymbol{\eta}_1 \tag{35}$$

In the time invariant case the output stabilization of (34)-(35) is equivalent to the full state feedback stabilization of

$$\widehat{\boldsymbol{\eta}}_1' = (\boldsymbol{\Theta}_1^T + \mathbf{A}_1^T)\widehat{\boldsymbol{\eta}}_1 + (\mathbf{H}\mathbf{U}_1)^T\mathbf{u} \tag{36}$$

However, this is not true for arbitrary linear time varying systems since the stability of $\mathbf{z}' = \mathbf{D}(t)\mathbf{z}$ is independent of the stability of $\mathbf{z}' = \mathbf{D}^T(t)\mathbf{z}$. A simple example is $\mathbf{D}(t) = \begin{bmatrix}-4 & e^{2t}\\ 0 & -1\end{bmatrix}$. Define $\mathbf{z}' = \mathbf{D}\mathbf{z}$ to be *symmetrically asymptotically stable* if $\mathbf{z}' = \mathbf{D}\mathbf{z}$ and $\mathbf{z}' = \mathbf{D}^T\mathbf{z}$ are both asymptotically stable. There are numerous conditions in the literature which guarantee symmetric asymptotic stability for linear time varying systems such as Lemma 6.2 of (Middleton *et al.*, 1988). The problem of determining which of these results is the most useful for the algorithms being developed here is still under investigation. Stabilizing feedback for uniformly controllable systems is discussed in (Ikeda *et al.*, 1972). The next theorem summarizes this discussion.

Theorem 1. Suppose that (1) is a solvable DAE with known smooth coefficients. Let $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{g}}, \widehat{\mathbf{G}}, \widehat{\mathbf{h}}$, $\widehat{\mathbf{v}}$ be from the explicit representation (5)-(6). Let $\mathbf{\Theta}_1, \mathbf{A}_1, \mathbf{U}_1$ be defined by (21), (28). Suppose (36) is uniformly controllable. Let $\overline{\mathbf{K}}^T$ be a stabilizing feedback and suppose that the closed loop system is symmetrically asymptotically stable. Let $\widehat{\mathbf{L}} = \mathbf{U}_1\overline{\mathbf{K}}$. Then (8)-(9) is an observer for (1)-(2).

4. NO CONSTRAINT APPROACH

This section briefly discusses the importance of including the constraints in the observer. One could try to ignore the constraints (6) and work with (5) and (7)

$$\mathbf{x}' = \widehat{\mathbf{A}}\mathbf{x} + \widehat{\mathbf{B}}\mathbf{v} + \widehat{\mathbf{g}} \quad (37)$$
$$\mathbf{y} = \mathbf{H}(t)\mathbf{x} \quad (38)$$

This appears to be a simpler problem. However, it is possible that the extra dynamics might be neither observable nor physical.

Example 1. As a simple example of (5)-(7) consider the DAE

$$x_1' = x_1 + x_2 + v \quad (39)$$
$$x_2' = x_2 \quad (40)$$
$$0 = x_2 \quad (41)$$
$$y = x_1 \quad (42)$$

For this problem it is easy to stabilize the error equation on the solution manifold ($x_2 = 0$) with feedback. However, the system

$$x_1' = x_1 + x_2 + v \quad (43)$$
$$x_2' = x_2 \quad (44)$$
$$y = x_1 \quad (45)$$

cannot be stabilized since the x_2 dynamics are uncontrollable.

5. INTEGRATION OF THE OBSERVER

(8)-(9) is an overdetermined DAE and thus the standard DAE integrators may have problems. Some overdetermined DAE integrators exist such as ODASSL which was developed for mechanics problems (Fuhrer and Leimkuhler, 1991).

One option would be to follow Gear (Brenan *et al.*, 1996) and introduce a dummy multiplier to get the DAE observer

$$\widehat{\mathbf{x}}' = \widehat{\mathbf{A}}\widehat{\mathbf{x}} + \widehat{\mathbf{B}}v + \widehat{\mathbf{g}} + \widehat{\mathbf{L}}(\mathbf{y} - \mathbf{H}\widehat{\mathbf{x}}) + \widehat{\mathbf{G}}^T\boldsymbol{\mu} \quad (46)$$
$$\mathbf{0} = \widehat{\mathbf{G}}(t)\widehat{\mathbf{x}} + \widehat{\mathbf{h}} \quad (47)$$
$$\mathbf{w} = \widehat{\mathbf{x}} \quad (48)$$

This is in the form of (3)-(4) with $\mathbf{z} = \begin{bmatrix} \widehat{\mathbf{x}} \\ \boldsymbol{\mu} \end{bmatrix}$, $\mathbf{P} = [\mathbf{I} \quad \mathbf{0}]$, $\mathbf{K} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. The DAE to be integrated, (46)-(47) is a semi-explicit index two DAE in $(\widehat{\mathbf{x}}, \boldsymbol{\mu})$ as long as $\widehat{\mathbf{G}}$ has full row rank. Thus this system can be integrated with a variety of BDF and implicit Runge Kutta methods (Brenan *et al.*, 1996).

Note that the value of $\boldsymbol{\mu}$ is not important. Replacing $\boldsymbol{\mu}$ by $\boldsymbol{\lambda}'$ in (46) gives an index one DAE. However, the solutions of interest are those for which $\boldsymbol{\lambda} = \mathbf{0}$ so that $\boldsymbol{\lambda}$ needs to approach zero. This leads to considering the following observer

$$\widehat{\mathbf{x}}' = \widehat{\mathbf{A}}\widehat{\mathbf{x}} + \widehat{\mathbf{B}}\mathbf{v} + \widehat{\mathbf{g}} + \widehat{\mathbf{L}}(\mathbf{y} - \mathbf{H}\widehat{\mathbf{x}}) + \widehat{\mathbf{G}}^T\boldsymbol{\lambda}' + \mathbf{\Gamma}\boldsymbol{\lambda} \quad (49)$$
$$\mathbf{0} = \widehat{\mathbf{G}}(t)\widehat{\mathbf{x}} + \widehat{\mathbf{h}} \quad (50)$$
$$\mathbf{w} = \widehat{\mathbf{x}} \quad (51)$$

This is in the form of (3)-(4) with $\mathbf{z} = \begin{bmatrix} \widehat{\mathbf{x}} \\ \boldsymbol{\lambda} \end{bmatrix}$, $\mathbf{P} = [\mathbf{I} \quad \mathbf{0}]$, and $\mathbf{K} = \begin{bmatrix} \mathbf{I} & -\widehat{\mathbf{G}}^T \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$.

Theorem 2. Assume that $\widehat{\mathbf{L}}$ is chosen so that the conditions of Theorem 1 hold. Thus the solution of the error equation (10)-(11) satisfies $\boldsymbol{\varepsilon} \to \mathbf{0}$. Let $\mathbf{\Gamma} = \alpha\widehat{\mathbf{G}}^T$ with $\alpha > 0$. Then (49)-(51) is an observer for (1)-(2). Furthermore $\boldsymbol{\lambda} = \boldsymbol{\lambda}_0 e^{-\alpha t}$.

Proof. (49)-(50) is an index one system whose solution manifold is determined by the constraints (50). Thus it suffices to prove that $\boldsymbol{\varepsilon} \to \mathbf{0}$. If $\widehat{\mathbf{x}}, \boldsymbol{\lambda}$ are given by (49)-(50), then $\boldsymbol{\varepsilon}, \boldsymbol{\lambda}$ satisfy

$$\boldsymbol{\varepsilon}' = \widehat{\mathbf{A}}\boldsymbol{\varepsilon} - \widehat{\mathbf{L}}\mathbf{H}\boldsymbol{\varepsilon} + \widehat{\mathbf{G}}^T\boldsymbol{\lambda}' + \alpha\widehat{\mathbf{G}}^T\boldsymbol{\lambda} \quad (52)$$
$$\mathbf{0} = \widehat{\mathbf{G}}\boldsymbol{\varepsilon} \quad (53)$$

Such an observer could be implemented by just integrating (52)-(53). However, in the proof it is helpful to assume that $\widehat{\mathbf{G}}$ is smooth. Differentiating (53) and then solving for $\boldsymbol{\lambda}'$ gives

$$\begin{aligned}\boldsymbol{\varepsilon}' &= (\widehat{\mathbf{A}} - \widehat{\mathbf{L}}\mathbf{H})\boldsymbol{\varepsilon} + \alpha\widehat{\mathbf{G}}^T\boldsymbol{\lambda} \\ &\quad -\widehat{\mathbf{G}}^\dagger\left((\widehat{\mathbf{G}}' + \widehat{\mathbf{G}}\widehat{\mathbf{A}} - \widehat{\mathbf{G}}\widehat{\mathbf{L}}\mathbf{H})\boldsymbol{\varepsilon} + \alpha\widehat{\mathbf{G}}\widehat{\mathbf{G}}^T\boldsymbol{\lambda}\right) \\ &= (\widehat{\mathbf{A}} - \widehat{\mathbf{L}}\mathbf{H})\boldsymbol{\varepsilon} - \widehat{\mathbf{G}}^\dagger(\widehat{\mathbf{G}}' + \widehat{\mathbf{G}}\widehat{\mathbf{A}} - \widehat{\mathbf{G}}\widehat{\mathbf{L}}\mathbf{H})\boldsymbol{\varepsilon} \quad (54)\end{aligned}$$

$$\boldsymbol{\lambda}' = -(\widehat{\mathbf{G}}\widehat{\mathbf{G}}^T)^{-1}((\widehat{\mathbf{G}}' + \widehat{\mathbf{G}}\widehat{\mathbf{A}} - \widehat{\mathbf{G}}\widehat{\mathbf{L}}\mathbf{H})\boldsymbol{\varepsilon} + \alpha\widehat{\mathbf{G}}\widehat{\mathbf{G}}^T\boldsymbol{\lambda}) \tag{55}$$

where $\widehat{\mathbf{G}}^\dagger = \widehat{\mathbf{G}}^T(\widehat{\mathbf{G}}\widehat{\mathbf{G}}^T)^{-1}$. The only solutions of (54)-(55) of interest are those which are also solutions of (52)-(53). That is, solutions where $\widehat{\mathbf{P}}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}$. Let $\overline{\mathbf{P}}$ be the solution of

$$\overline{\mathbf{P}}' = \widehat{\mathbf{A}}\overline{\mathbf{P}} - \overline{\mathbf{P}}\widehat{\mathbf{A}}, \quad \overline{\mathbf{P}}(t_0) = \widehat{\mathbf{P}}(t_0)$$

Then from (Terrell, 1994), $\overline{\mathbf{P}}$ is a projection for all t and $\overline{\mathbf{P}}$ commutes with the operator $\frac{d}{dt} - \widehat{\mathbf{A}}$. Furthermore, $\widehat{\mathbf{P}}$ and $\overline{\mathbf{P}}$ have the same ranges since the range of $\widehat{\mathbf{P}}$ is invariant under the differential equation $\mathbf{x}' = \widehat{\mathbf{A}}\mathbf{x}$. Thus $\widehat{\mathbf{G}}\overline{\mathbf{P}} = \mathbf{0}$ and $\widehat{\mathbf{G}}'\overline{\mathbf{P}} + \widehat{\mathbf{G}}\overline{\mathbf{P}}' = \mathbf{0}$. Hence

$$\widehat{\mathbf{G}}'\overline{\mathbf{P}} = -\widehat{\mathbf{G}}\overline{\mathbf{P}}' = -\widehat{\mathbf{G}}[\widehat{\mathbf{A}}\overline{\mathbf{P}} - \overline{\mathbf{P}}\widehat{\mathbf{A}}] = -\widehat{\mathbf{G}}\widehat{\mathbf{A}}\overline{\mathbf{P}}$$

so that $\widehat{\mathbf{G}}'\boldsymbol{\varepsilon} = -\widehat{\mathbf{G}}\widehat{\mathbf{A}}\boldsymbol{\varepsilon}$. By construction $\widehat{\mathbf{P}}\widehat{\mathbf{L}}\mathbf{H}\widehat{\mathbf{P}} = \widehat{\mathbf{L}}\mathbf{H}\widehat{\mathbf{P}}$ so that $\overline{\mathbf{P}}\widehat{\mathbf{L}}\mathbf{H}\overline{\mathbf{P}} = \widehat{\mathbf{L}}\mathbf{H}\overline{\mathbf{P}}$. Hence $\widehat{\mathbf{G}}\widehat{\mathbf{L}}\mathbf{H}\boldsymbol{\varepsilon} = \widehat{\mathbf{G}}\widehat{\mathbf{L}}\mathbf{H}\overline{\mathbf{P}}\boldsymbol{\varepsilon} = \widehat{\mathbf{G}}\overline{\mathbf{P}}\widehat{\mathbf{L}}\mathbf{H}\overline{\mathbf{P}}\boldsymbol{\varepsilon} = \mathbf{0}$. Thus (54)-(55) simplifies to

$$\boldsymbol{\varepsilon}' = (\widehat{\mathbf{A}} - \widehat{\mathbf{L}}\mathbf{H})\boldsymbol{\varepsilon} \tag{56}$$

$$\boldsymbol{\lambda}' = -\alpha\boldsymbol{\lambda} \tag{57}$$

and the result follows. □

6. COMPUTATION OF NEEDED ARRAYS

This section will summarize some facts concerning the computation of the observer coefficients from (Campbell, 1987; Campbell, 1992). Suppose the DAE is

$$\mathbf{E}(t)\mathbf{x}'(t) + \mathbf{F}(t)\mathbf{x}(t) = \mathbf{b}(t) \tag{58}$$

where $\mathbf{b}(t) = \mathbf{B}(t)\mathbf{u}(t)$ if there is a control.

Definition 1. The system $\mathbf{E}(t)\mathbf{x}' + \mathbf{F}(t)\mathbf{x} = \mathbf{b}(t)$ is *solvable* on the interval $\mathcal{I}$ if

(1) For every sufficiently smooth $\mathbf{b}$ on $\mathcal{I}$, there is a solution to the descriptor system.
(2) Solutions are defined on all of $\mathcal{I}$.
(3) Solutions are uniquely determined by their values at any t_0 in $\mathcal{I}$.

Differentiating the equation (58) j times gives the system of equations

$$\begin{bmatrix} \boldsymbol{\mathcal{F}}_j & \boldsymbol{\mathcal{E}}_j \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_j \end{bmatrix} = \mathbf{b}_j \tag{59}$$

where

$$\boldsymbol{\mathcal{F}}_j = \begin{bmatrix} \mathbf{F} \\ \mathbf{F}' \\ \vdots \\ \mathbf{F}^{(j)} \end{bmatrix}, \mathbf{b}_j = \begin{bmatrix} \mathbf{b} \\ \mathbf{b}' \\ \vdots \\ \mathbf{b}^{(j)} \end{bmatrix}, \mathbf{x}_j = \begin{bmatrix} \mathbf{x}' \\ \vdots \\ \mathbf{x}^{(j+1)} \end{bmatrix}$$

$$\boldsymbol{\mathcal{E}}_j = \begin{bmatrix} \mathbf{E} & \mathbf{0} & \cdot & \cdot & \mathbf{0} \\ \mathbf{E}' + \mathbf{F} & \mathbf{E} & \mathbf{0} & \cdot & \cdot \\ \mathbf{E}'' + 2\mathbf{F}' & 2\mathbf{E}' + \mathbf{F} & \mathbf{E} & \ddots & \cdot \\ * & * & * & \ddots & \mathbf{0} \\ \mathbf{E}^{(j)} + j\mathbf{F}^{(j-1)} & * & * & * & \mathbf{E} \end{bmatrix}$$

Theorem 3. Suppose that (58) is solvable on the interval $\mathcal{I}$ and that $\mathbf{E}, \mathbf{F}$ are $2n$-times differentiable. Then for $\overline{n} = n + 1$

$$\boldsymbol{\mathcal{E}}_{\overline{n}} \text{ has constant rank on } \mathcal{I} \tag{60}$$

$$\boldsymbol{\mathcal{E}}_{\overline{n}} \text{ is 1-full with respect to } \mathbf{x}' \tag{61}$$

$$[\boldsymbol{\mathcal{F}}_i \; \boldsymbol{\mathcal{E}}_i] \text{ has full row rank for } 1 \le i \le \overline{n} \tag{62}$$

If the coefficients $\mathbf{E}, \mathbf{F}$ are infinitely differentiable, then Theorem 3 provides sufficient as well as necessary conditions for solvability. If (62) holds, and $\mu + 1$ is the smallest value of $\overline{n}$ that satisfies the conditions (60), (61) of Theorem 3, then μ is called the *index* ν of the descriptor system (58). For linear time invariant descriptor systems, the index is the same as the index of the pencil $\lambda\mathbf{E} + \mathbf{F}$. However, for time varying solvable descriptor systems, the pencil $\lambda\mathbf{E} + \mathbf{F}$ need not be regular, and if the pencil is regular its index need not be that of the descriptor system.

Theorem 3 is important since it assures us that if the descriptor system (58) is solvable, then $\boldsymbol{\mathcal{E}}_j$ will have constant rank even if $\mathbf{E}$ does not. Thus a computation concerning $\boldsymbol{\mathcal{E}}_j$ can be well conditioned.

Lemma 1. Suppose that (58) is a solvable index ν descriptor system. Then for any $\ell \ge 0$, the row echelon form of $[\boldsymbol{\mathcal{E}}_{\nu+\ell}|\boldsymbol{\mathcal{F}}_{\nu+\ell}|\mathbf{b}_{\nu+\ell}]$ is

$$\left[\begin{array}{cc|c|c} \mathbf{I}_{n(\ell+1)\times n(\ell+1)} & \mathbf{0} & \tilde{\mathbf{Q}}_1 & \tilde{\mathbf{b}}_1 \\ \mathbf{0} & \mathbf{R} & \tilde{\mathbf{Q}}_2 & \tilde{\mathbf{b}}_2 \\ \mathbf{0}_{\rho\times n(\ell+1)} & \mathbf{0} & \boldsymbol{\mathcal{M}} & \tilde{\mathbf{b}}_3 \end{array}\right] \tag{63}$$

where $\mathbf{R}$ and $\boldsymbol{\mathcal{M}}$ have full row rank. Furthermore, the solutions of $\boldsymbol{\mathcal{M}}\mathbf{x} = \tilde{\mathbf{b}}_3$ are independent of ℓ.

Suppose that (58) is solvable and index ν. Performing orthogonal row operations on $[\boldsymbol{\mathcal{E}}_\nu \; \boldsymbol{\mathcal{F}}_\nu|\mathbf{b}_\nu]$ yields

$$\left[\begin{array}{cc|c|c} \widehat{\mathbf{Q}}_1 & \mathbf{0} & \widehat{\mathbf{Q}}_2 & \overline{\mathbf{b}}_1 \\ \mathbf{0} & \mathbf{S} & \widehat{\mathbf{Q}}_3 & \overline{\mathbf{b}}_2 \\ \mathbf{0} & \mathbf{0} & \widehat{\mathbf{G}} & -\widehat{\mathbf{h}} \end{array}\right] \tag{64}$$

where $\mathbf{S}, \widehat{\mathbf{G}}$ are full row rank, $\widehat{\mathbf{G}}$ is $\rho \times n$, and $\widehat{\mathbf{Q}}_1$ is $n \times n$ and invertible. The equation $\widehat{\mathbf{G}}\mathbf{x} = \widehat{\mathbf{h}}$ determines the solution manifold of (58) at time t. In addition,

$$\widehat{\mathbf{A}} = -\widehat{\mathbf{Q}}_1^{-1}\widehat{\mathbf{Q}}_2, \; \widehat{\mathbf{g}} = \widehat{\mathbf{Q}}_1^{-1}\overline{\mathbf{b}}_1 \tag{65}$$

Computed this way, $\widehat{\mathbf{A}}$ can have discontinuities where the pivoting strategy changes. $\widehat{\mathbf{A}}, \widehat{\mathbf{g}}$ in (5) can be computed smoothly by noting that $\boldsymbol{\mathcal{E}}_j$ is constant rank and thus its Moore Penrose inverse is continuous. Then

$$\boldsymbol{\mathcal{E}}_j^\dagger \boldsymbol{\mathcal{F}}_j = \begin{bmatrix} \widehat{\mathbf{A}} \\ * \end{bmatrix}, \ \boldsymbol{\mathcal{E}}_j^\dagger \mathbf{b}_j = \begin{bmatrix} \widehat{\mathbf{g}} \\ * \end{bmatrix}$$

The computation of $\widehat{\mathbf{L}}$ requires the quantities $\mathbf{U}_1, \mathbf{A}_1$ and $\boldsymbol{\Theta}_1 = -\mathbf{U}_1^T \mathbf{U}_1'$. Thus $\mathbf{U}_1, \mathbf{A}_1, \mathbf{U}_1'$ are needed. One could try to work with $\widehat{\mathbf{G}}$. However, $\boldsymbol{\mathcal{E}}_j$ and $\boldsymbol{\mathcal{E}}_j'$ are readily available. Thus the following approach can be used.

By assumption $\boldsymbol{\mathcal{E}}_j$ has constant rank. From (Kunkel and Mehrmann, 1991) there is a smooth $\mathbf{W}(t)$ such that

$$\mathbf{W}\boldsymbol{\mathcal{E}}_j = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix} \boldsymbol{\mathcal{E}}_j = \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{R}$ has full row rank. Numerical algorithms for computation of $\mathbf{W}$ and $\mathbf{W}'$ at a given t are in (Kunkel and Mehrmann, 1991). Let $\widehat{\mathbf{G}} = \mathbf{W}_2 \boldsymbol{\mathcal{F}}_j$. $\boldsymbol{\mathcal{F}}_j'$ can be computed by differentiating the known $\mathbf{F}(t)$ one more time. $\mathbf{W}_2'$ is computable by the algorithm in (Kunkel and Mehrmann, 1991). Thus

$$\widehat{\mathbf{G}}' = \mathbf{W}_2' \boldsymbol{\mathcal{F}}_j + \mathbf{W}_2 \boldsymbol{\mathcal{F}}_j'$$

is computable. Note that $\widehat{\mathbf{P}} = \mathbf{I} - \widehat{\mathbf{G}}^\dagger \widehat{\mathbf{G}}$. By a slight modification the arguments in (Kunkel and Mehrmann, 1991), it follows that

$$\widehat{\mathbf{P}}' = -(\widehat{\mathbf{G}}^\dagger \widehat{\mathbf{G}}' \widetilde{\mathbf{P}}) - (\widehat{\mathbf{G}}^\dagger \widehat{\mathbf{G}}' \widetilde{\mathbf{P}})^T$$

so that $\widehat{\mathbf{P}}'$ is computable.

Once $\widehat{\mathbf{P}}, \widehat{\mathbf{P}}'$ are known, computation of $\mathbf{U}, \mathbf{U}'$ for the $\mathbf{U}$ in (12) can be done by using the analytic singular value decomposition in (Kunkel and Mehrmann, 1991).

Thus all the needed quantities are computable. Before such an observer can be utilized, of course, it is necessary to develop efficient algorithms. Also the computation of $\overline{\mathbf{K}}$ must be meshed with the computation of the other quantities. These technical issues will be discussed elsewhere.

7. CONCLUSION

This paper has described a procedure for computing an observer for a general linear time varying solvable plant. It has been shown how each of the terms is computable.

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