AUXILIARY SIGNAL DESIGN FOR MULTI-MODEL IDENTIFICATION IN SYSTEMS WITH MULTIPLE DELAYS

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Abstract

In an active approach for model detection and its use in failure detection, an auxiliary control is applied in order to assist in model identification. Recently an active approach for robust multi-model identification and failure detection in the presence of bounded energy noise over possibly short time intervals has been introduced. This paper begins the examination of the extension of the original design procedure to problems with several delays. The original infinite dimensional delay problem will be approximated by a finite dimensional non-delayed system. A number of approximation schemes for systems with delays have been developed in the literature, however, their use in this setting is new. In this paper we shall present computational tests comparing two different approximation approaches. One is based on a discretization and the other on a spline approximation approach.

1 Introduction

Model detection and its use in failure detection can be done by an active or a passive approach. In a passive approach various outputs are monitored. In an active approach an auxiliary control is applied in order to assist in model identification. Sometimes failure detection is impossible without the use of an auxiliary signal. An example involving delays can be found in [5].

Recently an approach for robust multi-model identification and failure detection in the presence of bounded energy noise over possibly short time intervals has been introduced. This active approach [3, 4, 9, 10] has been developed for systems without delays. Since many practical systems have delays it is important to be able to carry out this type of design for systems with delays. A key part of this procedure is the formulation and solution of a nonstandard optimization problem. In the very special case when there is a single constant delay, and the detection horizon is an integer multiple of this delay, the previous procedure can be extended to the delay case using the well known method of steps. This, along with some related computational issues is discussed in [5]. While useful for special problems and also as a truth model for more sophisticated algorithms, the approach of [5] cannot be applied to many problems.

This paper begins the examination of the extension of the original design procedure to problems with several delays. The inclusion of general variable delays or multiple delays in the models being identified results in a problem which is intrinsically infinite dimensional and requires additional theoretical and numerical results. We are interested in the case where there are multiple delays and the number of delays and their values can vary from one model to another.

The approach we will follow will be to approximate the infinite dimensional delay problem with a finite dimensional non-delayed system. A number of approximation schemes for delayed systems have been developed in the literature. However, their use in our setting is new. In particular it is not clear which approximations will work best since several new issues are involved including how the structure of the approximation interacts with the numerical efficiency of the optimization algorithm and the requirement for software which must be able to handle high dimensional problems. In our case we use a direct transcription code called SOCS (Sparse Optimal Control Software) [1, 2], however, other industrial scale optimization packages could be used. The approximation problem we consider here is different from that of many approximation problems in that what is of primary importance is not the accuracy of the state but the suitability of the auxiliary signal that is being designed. What is desired are coarse problem approximations, so that the problem size is smaller, but which are fine enough to enable one to get a sufficiently good auxiliary signal.

2 Procedure without delays

Space prohibits a full discussion of the auxiliary signal design procedure. This can be found in [4]. Our emphasis in this paper is on issues specific to the solution of delay systems. However, this requires a quick summary of the procedure without delays. This procedure can be applied to problems with more than two models and to a number of variations but we focus on the simplest case in this paper.

The true system model is supposed to be one of the following two linear models defined on $[0, \omega]$:

$$x_i' = A_i x_i + B_i v + M_i \nu_i \quad \text{(1a)}$$
$$y = C_i x_i + N_i \nu_i \quad \text{(1b)}$$

for $i = 1$ and 2. Here v is the auxiliary signal, y is the output and the ν_i's represent perturbations, noises and unmeasured inputs. A_i, B_i, C_i, M_i, N_i are matrices, which can be time varying, of appropriate dimensions. The states x_i of the two models need not have the same dimensions. The same is true for the ν_i. It is assumed that v, ν_i are in $L^2[0, \omega] = L^2$. Then x_i, y will also be in L^2. To avoid confusion we let $|z|$ denote the usual Euclidian norm of a vector and $\|z\|$ denote the L^2 norm of a vector function. We assume that C_i and N_i have full row rank.

The noise in model i is measured by

$$\mathcal{S}_i(x_i(0), \nu_i) = x_i(0)^T Q_i x_i(0) + \int_0^\omega |\nu_i(t)|^2 dt \quad (2)$$

where Q_i is positive semidefinite. Note that if $Q_i = 0$, it means that there is no prior information on x_i. It is assumed that the noise is bounded so that

$$\mathcal{S}_i(x_i(0), \nu_i) < 1 \text{ for } i = 1, 2. \quad (3)$$

The auxiliary signal v is not proper if there exist x_0, x_1, ν_0, ν_1, and y satisfying (1a), (1b) and (3) both for $i = 1$ and $i = 2$. The auxiliary signal v is called proper otherwise.

If $\mathcal{A}_i(v)$ is the set of outputs for model i with noise satisfying (3), then properness can also be expressed as $\mathcal{A}_1(v) \cap \mathcal{A}_2(v) = \emptyset$.

We want to compute a minimal norm proper test function. This is

$$\min \|v\|$$

such that

$$\max\{\mathcal{S}_1(x_1(0), \nu_1), \mathcal{S}_2(x_2(0), \nu_2)\} \geq 1 \quad (4)$$

subject to (1).

To convert this min-max problem to one that is more computationally tractable we proceed as follows. The function

$$J_v(\beta) = \inf_{x_i, \nu_i, y} \beta \mathcal{S}_1(x_1(0), \nu_1) + (1-\beta) \mathcal{S}_2(x_2(0), \nu_2)$$
$$\text{subject to (1a)-(1b)} \quad (5)$$

for $0 \leq \beta \leq 1$ is the auxiliary cost function associated with problem (1a)-(1b). It can be shown that an auxiliary signal v is proper if and only if $J_v(\beta) \geq 1$ for some $0 < \beta < 1$. Thus (4) can be replaced by

$$J_v(\beta) \geq 1. \quad (6)$$

Then [4] proceeds by setting up the optimization problem of minimizing $\|v\|$ but replaces (6) by the necessary conditions for the problem that defines $J_v(\beta)$. Later when we refer to "the necessary conditions" we mean these necessary conditions.

3 Basic Problem

In this paper we consider the same problem as in Section 2 except that we allow for delays. We assume that we have m possible models of the form

$$x_i'(t) = A_i x_i(t) + \sum_{j=1}^{r_i} G_{ij} x_i(t - \tau_{i,j}) + B_i v(t) + M_i \nu_i(t) \quad (7a)$$

$$y(t) = C_i x_i(t) + N_i \nu_i(t) \quad (7b)$$

$$x_i(t) = \phi_i(t),\ -h_i \le t < 0,\ x_i(0) = x_{i,0} \quad (7c)$$

for $i = 1, \ldots, m$ where v is an auxiliary signal to be determined, y is the observed output, and $h_i = \tau_{i,r} > \tau_{i,r-1} > \cdots \tau_{i,1} > 0$. Let $h = \max_i h_i$. Note that v, y are independent of i and are considered known since v will be precomputed and y is measured online during the detection test. The delays and coefficients may vary from model to model. For simplicity of discussion we shall take the coefficient matrices as constants. However, the discussion goes through with only the obvious changes if the coefficients are continuous functions of t. We assume to not know ϕ_i and thus it is a type of disturbance or noise as are the ν_i. Finally we suppose that the N_i are full row rank and that the detection horizon is $[0,\ \omega]$ with $\omega > h$. This assumption on N_i is not restrictive since it means we are handling the worse case when all output channels are noisy.

Notice in the system (7), as is typical with delay systems, there are two initial conditions $x_{i,0}$ and ϕ_i and they are considered to be independent.

Detection is to be carried out over a finite interval $[0, \omega]$. We suppose that the disturbances $\phi_i, \nu_i, x_i(0)$ satisfy

$$\mathcal{S}_i(\phi_i, \nu_i, x_{i,0}) = \int_{-h_i}^{0} \|\phi_i\|^2 dt + x_i(0)^T Q_i x_i(0) + \int_0^{\omega} \|\nu_i\|^2 dt \le 1 \quad (8)$$

where Q is positive semi-definite. No weight is necessary on the second integral since such a weight can be accommodated by redefining the N_i, M_i.

Let $\mathcal{A}_i(v)$ be the set of outputs y for (7) given (8) holds. Then an auxiliary signal v is proper if it leads to disjoint output sets $\mathcal{A}_i(v)$. That is, $\mathcal{A}_i(v) \cap \mathcal{A}_j(v) = \emptyset$ if $i \neq j$.

The first step is to calculate a minimal energy proper auxiliary signal. Once this auxiliary signal is determined one needs a test to determine which model is correct. This is done by using separating hyperplanes in [3, 4, 9]. As in these earlier papers we are interested in the possibility of rapid detection so that the time intervals are finite and allowed to be "small." Also the computation of v and the functions needed in the hyperplane test may be computed offline. Only the application of the hyperplane test needs to be done online.

While more than two models are easily handled if the approach of [4] is used, we will usually work with just two models to simplify the notation and discussion.

The auxiliary signal v is proper if getting the same output from both models would require too much noise. That is, if (7) holds for $i = 1, 2$, then (8) is violated for at least one of $i = 1, 2$. Thus finding the minimum proper v is a constrained min-max problem with the max over (8) which is a max over a discrete set. Using theory this can be reformulated as a max-min over continuous variables. As mentioned earlier, in the approach of [4] which we extend in this paper, the inner min is replaced by its necessary conditions to create a new optimization problem for v. These necessary conditions turn out to have implications for our approximations as will be shown later.

3.1 Needed Theory

Let $\mathcal{L}g$ be the solution of the delay equation

$$x_i'(t) = A_i x_i(t) + \sum_{j=1}^{r_i} G_{ij} x_i(t - \tau_{i,j}) + g(t) \quad (9a)$$

$$x_i(t) = 0, \text{ for } -h_i \le t \le 0. \quad (9b)$$

Let $\Gamma\phi$ be the solution of

$$x_i'(t) = A_i x_i(t) + \sum_{j=1}^{r_i} G_{ij} x_i(t - \tau_{i,j}) \quad (10a)$$

$$x_i(t) = \phi(t), \text{ for } -h_i \le t < 0. \quad (10b)$$

Finally let Θ be the solution of

$$x_i'(t) = A_i x_i(t) + \sum_{j=1}^{r_i} G_{ij} x_i(t - \tau_{i,j}) \quad (11a)$$

$$x_i(t) = 0, \text{ for } -h_i \le t < 0, \; x_i(0) = I. \quad (11b)$$

Then the output in (7) can be written as

$$\begin{aligned} y &= C_i\mathcal{L}_iB_iv + C_i\mathcal{L}_iM_i\nu_i \\ &\quad + C_i\Gamma_i\phi_i + \Theta_i x_{i,0} + N_i\nu_i. \end{aligned} \quad (12)$$

The various operators are all bounded linear operators from different L^2 spaces into the same L^2. Thus $\mathcal{A}_i(v)$ is a convex set in L^2 translated by the vector $C_i\mathcal{L}_iB_iv$. We have that the set of ϕ_i, ν_i satisfying (8) are L^2 bounded. If $Q_i > 0$, then the convex set $\mathcal{A}_i(v)$ is bounded. If some or all Q_i are indefinite, then $\Theta_i x_{i,0}$ is a finite dimensional subspace. Thus we have that

Lemma 1 *Suppose that $Q_i > 0$ for $i = 1, \dots, m$. Then cv is proper for sufficiently large scalars c if and only if $C_i\mathcal{L}_iB_iv \neq C_j\mathcal{L}_jB_jv$ for $i \neq j$.*

Proof Notice that $C_i\mathcal{L}_iB_iv \neq C_j\mathcal{L}_jB_j$ is all that is required for a multiple of v to be proper for a comparison of model i to model j. But if v is proper, then any larger multiple is also proper. ■

There is a variation on this problem. In our failure detection application we can have the situation where the process is running and the test will be performed during operation. In this scenario while we do not know what $\phi_i, x_{i,0}$ are, we do know that the state is continuous. In this case we may assume that

$$\phi_i(0) = x_{i,0}. \quad (13)$$

We will not discuss this here other than pointing out that more is involved than just (13). The condition (13) holds only if some type of continuity is present. One way to do this would be to assume that ϕ is the output of some noise driven exogenous process. That will be discussed elsewhere.

4 Approximation by systems without delays

The method of steps used in [5] does not work well, or at all, if there are multiple delays. It also leads to very high dimensional problems if h is small relative to ω. We wish to consider the problems where there are not only multiple delays but a change in delay itself could be a source of failure. Suppose then that we have one of the models of the form (7). We shall temporarily suppress the subscript i to simplify the notation and describe the approximation process for one model. We can rewrite (7) as follows. Let $U(t,s) = x(t+s)$ for $0 \le t \le \omega, -h \le s \le 0$. Thus for a fixed t, $U(t,s)$ is a function in L^2 which is x on the interval $[t-h, \; t]$. To guarantee that U is the function we want we require U to satisfy the following well known partial differential equation

$$\begin{aligned} U_t(t,s) &= U_s(t,s) && (14a) \\ U_t(t,0) &= AU(t,0) + \sum_{j=1}^{r} G_jU(t,-\tau_j) \\ &\quad + Bv(t) + M\nu(t) && (14b) \\ U(0,s) &= \phi(s) \quad -h \le s < 0 && (14c) \\ U(0,0) &= x_0 && (14d) \\ y(t) &= CU(t,0) + N\nu(t) && (14e) \end{aligned}$$

There are a number of ways to approximate PDEs by ODEs. When applied to (14) we get an ODE model identification problem similar to that described in Section 2. However, there are a number of technical problems to be resolved in our setting including which methods work, the accuracy of approximation, and the relationship between optimization of the approximation and the original problem and how to guarantee detection of the original problem when using approximate problems to compute the test signal. We now turn to considering two types of approximations.

Different methods of approximation can lead to different models. Thus it is desirable to use the same approximation procedure in all models. Otherwise there is the danger that the test signal will be designed to detect differences due to the approximation methods rather than differences in the original models.

4.1 Use of differences

One method of approximating the PDE is the use of differences and the method of lines. This approach proceeds as follows. We temporarily suppress the i subscript in (14). We pick a mesh for $[-h, 0]$ of the form

$$-h = s_0 < s_1 < \cdots s_\rho = 0.$$

The one restriction on this mesh is that each $-\tau_j$ has to be a mesh point s_{m_j}. We suppose then that $-\tau_j = s_{m_j}$. The value of h and the delays may vary from model to model. Let

$$U_k(t) = U(t, s_k).$$

Then we get the following ordinary differential equation system

$$U_0'(t) = \sum_{h=0}^{2} \alpha_{0,h} U_h(t) \quad (15a)$$

$$U_k'(t) = \sum_{h=-1}^{1} \alpha_{k,h} U_{k+h}(t),\ 0 \le k < \rho - 1 \quad (15b)$$

$$U_\rho'(t) = AU_\rho(t) + \sum_{j=1}^{r} G_j U_{m_j}(t) + Bv(t) + M\nu(t) \quad (15c)$$

$$U_k(0) = \phi(s_k), \quad k < \rho \quad (15d)$$

$$U_\rho(0) = x_0 \quad (15e)$$

where (15b) and (15a) come from (14a) while (15c) comes from (14b). If we have $0 < k < \rho$ and let $\delta = s_{k+1} - s_k,\ \epsilon = s_k - s_{k-1}$, then we can take

$$\alpha_{k,1} = \frac{\epsilon}{\delta(\epsilon + \delta)}, \quad \alpha_{k,-1} = -\frac{\delta}{\epsilon(\epsilon + \delta)}, \quad (16)$$

$$\alpha_{k,0} = \frac{\epsilon}{\delta(\epsilon + \delta)} - \frac{\delta}{\epsilon(\epsilon + \delta)}. \quad (17)$$

This provides an approximation for the spatial derivative which is $O(\delta\epsilon)$. To get a similar accuracy for the U_0 equation we need to use a one sided approximation using two extra values. Let $\delta = s_1 - s_0,\ \epsilon = s_2 - s_1$. We can then take

$$\alpha_{0,0} = -\alpha_{0,2} - \alpha_{0,1}, \quad \alpha_{0,1} = \frac{\delta + \epsilon}{\delta\epsilon}, \quad (18)$$

$$\alpha_{0,2} = -\frac{\delta}{\epsilon(\delta + \epsilon)}. \quad (19)$$

Two things should be noticed. First we have that

$$a_{k,1} \le \frac{1}{\delta}, \quad a_{k,-1} \le \frac{1}{\epsilon}$$

so that the coefficients only grow as one over the mesh size. Secondly if $\epsilon = \delta$, then (17) simplifies to

$$a_{k,1} = \frac{1}{2\delta}, \quad a_{k,0} = 0, \quad a_{k,-1} = -\frac{1}{2\delta}.$$

This procedure has transformed the delay model equation to an approximate ordinary differential equation model. We also must transform the first term in the noise measure (8). Let $U_{i,j}$ be the j vector in the approximation for model i. Let $\gamma_{i,j}$ be the collocation coefficients for approximating integrals on $[-h_i, 0]$ using the grid points $s_{i,j}$. Then (8) becomes

$$\tilde{\mathcal{S}}_i(\phi_i, \nu_i, x_{i,0}) = \sum_{j=0}^{\rho_i - 1} \gamma_{i,j} \|U_{i,j}(0)\|^2 + \gamma_{i,\rho_i} \|\phi_i(s_{\rho_i})\|^2 + U_{i,\rho_i}(0)^T Q_i U_{i,\rho_i}(0) + \int_0^\omega \|\nu_i\|^2 dt \le 1. \quad (20)$$

Note that (20) is of the same type of noise measure studied previously for the undelayed problem. However, there is one important difference. In the optimization approach we use here, which is based on [4], a max min problem is solved by replacing the min by the necessary conditions for the min along with a quantity that returned the value of the min. In the non-delayed case this was a boundary value problem. However, in general, if in the cost, which is based on the noise bounds for the two models, there are terms that do not appear in any of the dynamic equations or constraints, then these necessary conditions have the additional requirement that these terms are zero.

For our problem here note that $\phi_i(0)$ appears only in the noise bound (20) and nowhere in the dynamics or initial conditions in (15). Accordingly, we have an additional necessary condition that

$$\phi_i(0) = 0 \quad (21)$$

so that instead of (20) we get

$$\tilde{\mathcal{S}}_i(\phi_i, \nu_i, x_{i,0}) =$$

$$\sum_{j=0}^{\rho_i-1} \gamma_{i,j}\|U_{i,j}(0)\|^2 \\ +U_{i,\rho_i}(0)^T Q_i U_{i,\rho_i}(0) + \int_0^\omega \|\nu_i\|^2 dt. \quad (22)$$

It is easy to see that if the condition (21) is not added then there is an incorrect solution of the optimization problem which has $v = 0$ and, in fact, we have seen this solution occur in computational tests which did not assume (21).

We shall use collocation schemes for which all the $\gamma_{i,j} > 0$. For example if we are using a Trapezoidal approximation, then we have (suppressing the i subscript on all terms)

$$\gamma_0 = \frac{s_1 - s_0}{2}, \quad \gamma_\rho = \frac{s_\rho - s_{\rho-1}}{2},$$

$$\gamma_j = \frac{s_{j+1} - s_{j-1}}{2} \text{ if } j \notin \{0, \rho\}. \quad (23)$$

Figure 1 from [6] shows the minimum energy test signal for a typical problem as the Q_i varies.

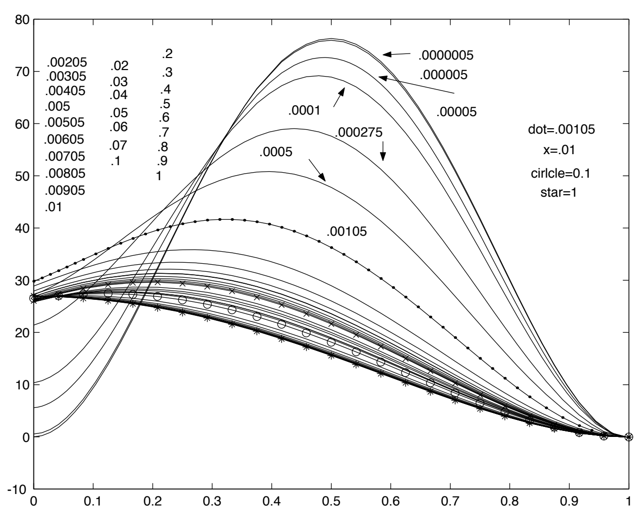

Figure 1: Minimum proper v on $[0, 1]$ for $0 \le Q_i \le 1$.

The important thing to notice is that Q_i had to be quite small before the v started to look like the $Q = 0$ case. Note that the quadrature always puts a weight on $\phi_i(0)$.

4.2 Spline approximation

An alternative approach to replacing the delay system with an ODE approximation is by approximating the various operators and functions from finite dimensional subspaces. The approach we consider is from [7, 8]. We follow here the notation and approach of [7]. We shall leave out much of the technical detail. It should be noted that this approach allows us to consider more complex models than (7a). In particular, one can include terms such as $\int_{-h_i}^0 F_i(\theta)x_i(t+\theta)d\theta$.

Pick N to be the partition size. Let t_k^N be a partition of $[-h, 0]$, $k = 0, \ldots, N$. Both ends are doubled by adding t_{-1}^N, t_{N+1}^N. $k = 0$ starts at the right end of $[-h, 0]$. Let $B_k^N(\theta)$ be the usual "hat" function. Note that except for $k = 0, N$, the area is one and for $k = 0, N$, the area is $\frac{1}{2}$. Thus if $g(t)$ is a function on $[-h, 0]$, we have that $\sum_{j=0}^N g(t_k^N)B_k^N(t)$ is a piecewise linear interpolation of $g(t)$ at the grid points. It is assumed that every delay of the original system falls into at most one of the subintervals. Note that, unlike with differences, the grid points do not have to hit the delay so we can use a uniform grid.

If initial data is $R^n \times L^2$, and $u \in L^2$, then solutions of (7a) are in $L^2(-h, T; R^n) \cap H^1(0, T; R^n)$. We take $\text{dom}(\mathcal{A}) = \{(\eta, \phi) \in Z | \phi \in H^1(-h, 0; R^n), \eta = \phi(0)\}$ and $\mathcal{A}(\phi) = \sum A_i\phi(\theta_i)$. H^1 is given the norm and inner product

$$< \phi, \psi >_{H_1} = < \phi(0), \psi(0) >_{R^n} + < \phi', \psi' >_{L^2} .$$

We define Ψ an isomorphism between $\text{dom}(\mathcal{A})$ and H^1 by $\Psi(\phi(0), \phi) = \phi$. Then $\Psi^{-1}(\phi) = (\phi(0), \phi)$. Other notation includes: $E_k^N = \aleph_{[t_k^N, t_{k-1}^N)}$, $k = 1, \ldots, N$ for the characteristic function of a subinterval; $\widehat{E}_0^N = (I_n, 0)$ for R^n; $\widehat{E}_k^N = (0, I_n)$ for the past; $W^N = span\{E_k^N I_n : 1 \le k \le N\}$ for the approximation subspace of $L^2[-r, 0]$; $Z^n = R^n \times W^N = span\{E_k^N I_n : 0 \le k \le N\}$ for the approximation subspace of Z; $X^N = span\{B_k^N I_n : 0 \le k \le N\}$ for the approximation subspace of H^1; and $Z_1^N = \Psi^{-1}X^N$ for the approximation subspace of $\text{dom}(\mathcal{A})$. In addition we have the basis matrices $E^N = (E_1^N I \cdots E_N^N I)$, $\widehat{E}^N = (\widehat{E}_0^N \cdots \widehat{E}_N^N)$, and $\widehat{B}^N = (B_0^N I \cdots B_N^N I)$.

Then $z = (\eta, \phi) \in Z^N$ can be written $(\eta, E^N a^N) =$

$\widehat{E}^N col(\eta, a^N)$ where a^N is a coordinate vector of $\phi \in W^N$ with respect to the basis. Similarly if $\phi \in X^N$, then $\phi = B^N b^N$, $b^N = col(b_0^N, \ldots, b_N^N)$.

Let

$$Q^N = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \frac{1}{2} & \frac{1}{2} & \ddots & \cdot & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \otimes I_n \qquad (24)$$

where the first matrix is $(N+1) \times (N+1)$. For $k = 0, \ldots, N$, let

$$D_k^N = L(B_k^N) = \sum_{j=0}^{c} A_j B_k^N(\theta_j). \qquad (25)$$

Also let

$$\begin{aligned} H^N &= \begin{bmatrix} D_0^N & \cdots & D_N^N \\ 0 & \cdots & 0 \end{bmatrix} \\ &+ \frac{N}{r} \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ o & \cdots & 0 & 1 & -1 \end{bmatrix} \otimes I_n \end{aligned} \qquad (26)$$

where the first matrix in the tensor product is $(N+1) \times (N+1)$ and the first matrix after the equality has lots of zero rows. Then

$$(Q^N)^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & \ddots & \cdot & \vdots \\ 1 & -2 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ (-1)^{N+1} & \cdots & 2 & -2 & 2 \end{bmatrix} \otimes I_n. \qquad (27)$$

B^N is a block column matrix whose only nonzero entry is the first one as is M^N. C^N is a block row matrix whose only nonzero entry is the first one which is C as is N^N.

Let

$$A^N = H^N (Q^N)^{-1} \qquad (28)$$

The differential equation is then

$$(a^N)'(t) = A^N a^N(t) + B^N v(t) + M^N \eta(t) \qquad (29)$$
$$y(t) = C^N a^N(t) + N^N \eta(t) \qquad (30)$$

5 Computational examples

In this section we shall work several computational examples. We consider two models of the form

$$x_i'(t) = A_i x_i(t) + G_i x_i(t-h) + B_i v + M_i \nu_{i,1} \quad (31a)$$
$$y = C_i x_i(t) + N_i \nu_{i,2} \quad (31b)$$

5.1 Computational Example 1

Our first example (32) will be used to illustrate a number of points. It is chosen since it is possible to reformulate this problem using the method of steps [5] and obtain the exact solution for comparison purposes.

$$x_1' = -2x_1 + x_1(t-1) + v + \nu_1 \quad (32a)$$
$$y = x_1 + \nu_2 \quad (32b)$$
$$x_2' = -3x_2 + x_2(t-1) + v + \nu_3 \quad (32c)$$
$$y = x_2 + \nu_4 \quad (32d)$$

We have solved this problem using steps, differences, and a spline approximation for a number of different choices of initial conditions and noise weightings. We suppose that detection is being done over $[0, 2]$. The two models differ in the decay rate.

Figure 2 shows the "true" minimal energy auxiliary signal v computed using steps and the auxiliary signals v_ρ gotten using differences with several values of ρ.

Observe that the approximation appears better for increasing ρ although on this problem the coarsest approximation using $\rho = 4$ already did quite well. The first column of Table 1 gives the norm of the test signal.

The question arises as to how good the minimal energy auxiliary signal from the approximation is. To examine this we took v_ρ and found the minimum value of c_ρ such that $c_\rho v_\rho$ was proper in the original system. This computation showed that for this example, not only were the v_ρ proper for the true problem but that $c_\rho v_\rho$ was close to minimal. Table 1 shows the norms of v, v_ρ and $c_\rho v_\rho$, for several values of ρ.

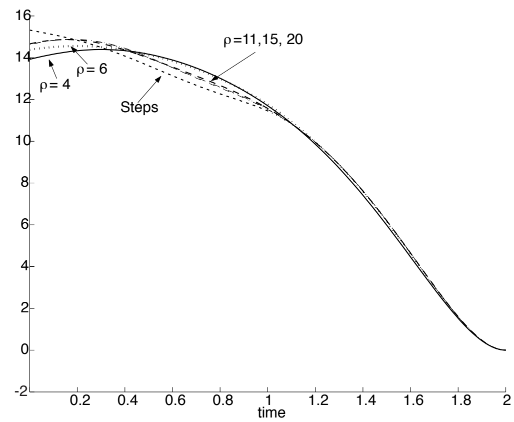

Figure 2: Minimal energy auxiliary signal v, and difference approximations v_4, v_6, v_{11}, v_{15}, v_{20} for Example 1.

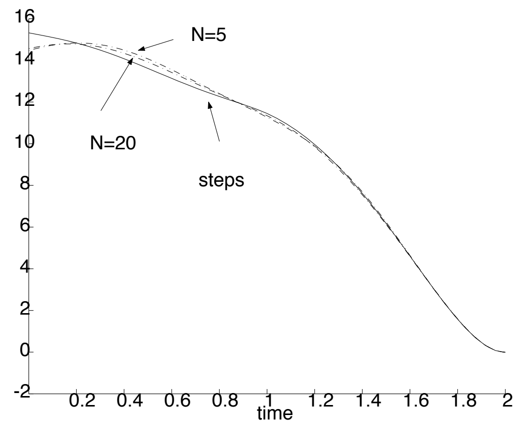

Figure 3: Auxiliary signals for Example 1 using splines for several N.

	$\|v_i\|$	c_i	$\|c_i v_i\|$
v_4	15.3796	0.9953	15.3071
v_6	15.3796	0.98699	15.3012
v_{11}	15.4598	0.98943	15.2965
v_{15}	15.4506		
v_{20}	15.4506		
v	15.2934	1	15.2934

Table 1: $\|v\|, \|v_i\|, c_i$ for Example 1 using differences with $\rho = 4, 6, 11, 15, 20$.

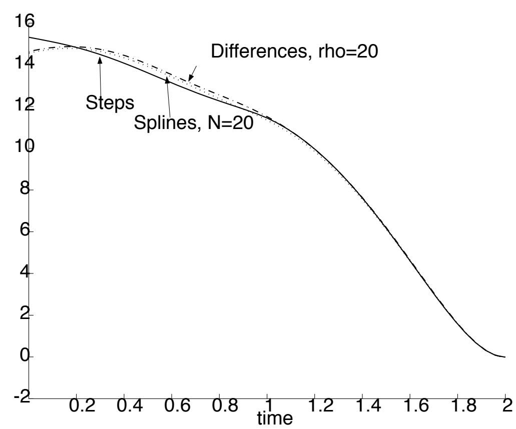

Figure 4: Differences and splines on Example 1 with $N = \rho = 20$ and the steps v.

We have also solved this problem using the spline approximations described earlier. The results are shown in Figure 3.

It is interesting to note that as ρ and N are increased the difference and spline approximations for v seem to converge to the same function. In fact, by $\rho = N = 20$ they are almost identical as shown in Figure 4.

The spline and difference approximation minimum energy auxiliary signals are differing slightly from those of the steps solution. However, the norm of the spline approximation is very close to that of the step solution as seen in Table 2.

We see that both the splines and differences do an excellent job of approximating and give us high quality auxiliary signals using low values of N and ρ. The source of the remaining difference between the approximate v and the steps v is not clear at this time. Notice that the difference in norms is quite small and the relative error shown in Table 2 is around 10^{-5}. This suggests that the difference is due to the fact that the answers from all three approaches are essentially equal in terms of the optimization problem and the remaining difference is due to how they approach the minimum. This is the most likely explanation. However, we will continue to examine this point. Graphically the spline and difference approximations appear to be equally good. The tables show that the spline approximations give a slightly better estimate of $\|v\|$.

	$\|v_N\|$	Rel. Error
N=3	15.49152	0.0129546
N=4	15.40969	0.0076039
N=5	15.37202	0.0051408
N=20	15.301224	0.0005113
v	15.2934	0

Table 2: Norm of spline approximation solutions for v for Example 1.

5.2 Computational Example 2

The next example illustrates a computational subtlety. Suppose that the two models are

$$
\begin{aligned}
x_1' &= -2x + x(t-1) + v + \nu_i && \text{(33a)}\\
y &= x + \nu_2 && \text{(33b)}\\
x' &= -2x + x(t-0.4) + v + \nu_3 && \text{(33c)}\\
y &= x + \nu_4 && \text{(33d)}
\end{aligned}
$$

so that the failure consists of a change in the delay interval. Suppose that we decide for convenience to use a delay interval of $[-1, 0]$ for both models. When we solve the problem for $\rho = 5$ or $N = 5$ we get the auxiliary signal in Figure 5.

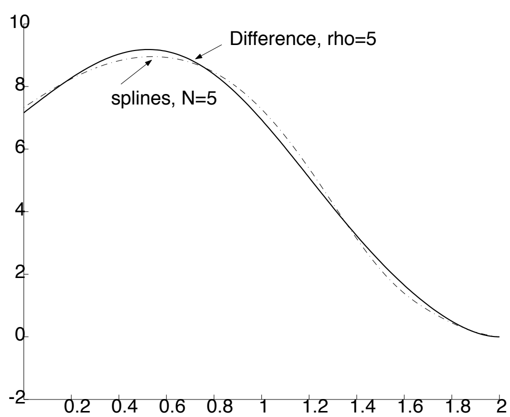

Figure 5: Minimal energy test signal for Example 2 using splines and differences with $h_1 = h_2 = 1$ and $N = \rho = 5$.

However, if we take $\rho \geq 11$ we begin to experience numerical difficulties. To understand what is happening we note that the approximate problem is a reasonable finite dimensional model. However, in the continuous problem the initial perturbation for model 2 on $[-1, -0.4]$ appears in the noise measure. Thus the correct formulation of the optimization version of our approach would require the initial perturbation to be zero on this interval. The failure to take this difference into account leads to a numerical breakdown as the width of the mesh used to partition the delay interval goes to zero. Thus, like with our handling of $\phi(0)$, one must keep the theoretical approach in mind and make sure that all the necessary conditions are included in the formulation.

5.3 Computational Example 3

For our final computational example we suppose that two delays are present and that the two models are

$$
\begin{aligned}
x_1' &= -2x_1 + x(t-1) + v + \nu_1 && \text{(34a)}\\
y &= x_1 + \nu_2 && \text{(34b)}\\
x_2' &= -2x_2 + 0.5x_2(t-1) \\
&\quad +0.5x_2(t-0.4) + v + \nu_3 && \text{(34c)}\\
y &= x_2 + \nu_4 && \text{(34d)}
\end{aligned}
$$

Here the failure consists of an additional shorter delay of the state. This type of change can occur if the delay in a feedback loop is altered, for example by failure of the hardware or software implementing the feedback. The minimum energy auxiliary signal found using difference approximations is given in Figure 6.

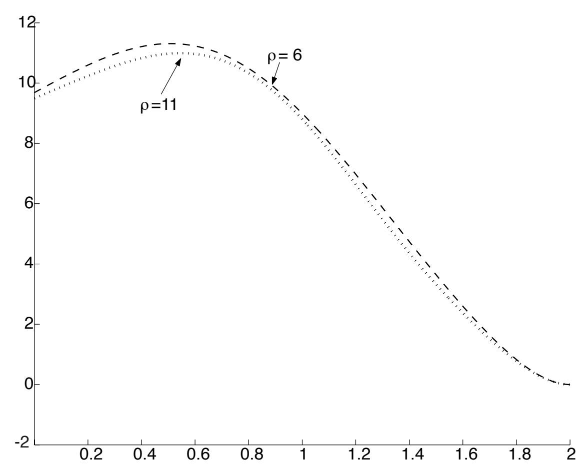

Figure 6: v_6 and v_{11} for Example 3.

The spline solution of Example 3 is almost identical as seen in Figure 7. It is interesting to note the differ-

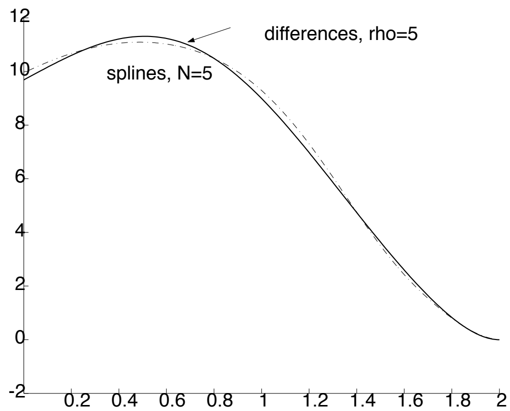

Figure 7: Spline and difference approximations with $N = \rho = 5$ on Example 3.

ence in shape between the auxiliary signals in Figure 6 and the auxiliary signals in Figure 2. In Example 3 where part of the difference in the two models is due to the additional delay of 0.4 we see that v acts most strongly in the interval $[0.4, 0.6]$.

6 Conclusion

This paper has shown how an auxiliary signal design approach for non-delay systems can be extended to delay systems by using finite dimensional approximations. This extension required modification of the noise measure in the approximation in order to get the correct solution of the finite dimensional problem. Computational examples suggest that high accuracy is not needed in the approximate system in order to get near optimal test signals. Further work is under way to examine what happens on more complex problems and to provide the needed analysis and error estimations for the proposed approach.

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