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# ON A GENERAL DESCRIPTOR LQ PROBLEM

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**Abstract** A discrete-time descriptor linear quadratic (LQ) optimization problem is considered. This problem can be considered as a generalization of many standard LQ problems including Kalman filters. A complete recursive solution based on a novel Riccati equation is presented. The limiting case when the size of the interval goes to infinity is studied.

## 1 INTRODUCTION

In this paper, we study a generalized linear least squares fit problem. The type of problem we consider here comes up often in the stochastic framework where the Kalman filtering, identification, or the likelihood computations can be formulated as LQ optimization problems [11, 12]. These LQ optimization problems don't have the standard structure encountered in control problems in that the associated dynamics equation is in descriptor form and rectangular. The reason is that the dynamics is obtained by putting together the dynamics of the original system, which may or may not be in descriptor form, and the observation equations. Of course, the original dynamics may also be in descriptor form [7, 6, 8, 10, 2, 4].

We use the method of dynamic programming (see for example [1]) to construct a recursive solution which turns out to be based on a generalized Riccati equation. We study the asymptotic properties of this equation and propose a method for the construction of its solution.

## 2 PROBLEM FORMULATION

We consider the problem

$$J(v) = \min \sum_{k=0}^{N-1} \nu(k)^T \nu(k) \tag{2.1}$$

subject to

$$E\xi(k+1) = F\xi(k) + G\nu(k) + Hv(k), \tag{2.2}$$

where $v$ is a known given sequence of vectors. $E$, $F$, $G$ and $H$ are matrices with appropriate dimensions.

We assume

$$\begin{pmatrix} sE - tF \end{pmatrix} \text{ has full column rank } \forall (s,t) \neq 0, \tag{2.3}$$

$$\begin{pmatrix} sE - tF & G \end{pmatrix} \text{ has full row rank } \forall (s,t) \neq 0. \tag{2.4}$$

Condition (2.3) is necessary for the uniqueness of the solution. If this condition is not satisfied, we can remove the part of $\xi$ which is arbitrary (the "unobservable subspace"). If condition (2.3) is satisfied and $N \geq n$, where $n$ is the size of the vector $\xi(k)$, then there is at most one solution to the problem.

Condition (2.4) guarantees the existence of solution for all $v$.

## 3 METHOD OF DYNAMIC PROGRAMMING

To construct the solution of problem (2.1) subject to constraints (2.2), we use the method of dynamic programming. Let $J_i(\xi(i))$ denote the past cost function:

$$J_i(\xi(i)) = \min \sum_{k=0}^{i-1} \nu(k)^T \nu(k) \tag{3.5}$$

subject to

$$\begin{aligned} E\xi(k+1) = F\xi(k) + G\nu(k) + Hv(k), \\ k = 0, \cdots, i-1. \end{aligned} \tag{3.6}$$

Note that $J_i$ depends on given constants $v$ but for simplicity of the notations, we do not explicitly express this dependence. Clearly,

$$J(v) = \min_{\xi(N)} J_N(\xi(N)). \tag{3.7}$$

The $J_i$'s can be constructed recursively as follows:

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**Lemma 3.1**

$$J_i(\xi(i)) = \left(s(i) - \begin{pmatrix}0\\F\end{pmatrix}\xi(i)\right)^T \Gamma(i)\left(s(i) - \begin{pmatrix}0\\F\end{pmatrix}\xi(i)\right) + \sum_{j=0}^{i-1} s(j)^T\Gamma(j)s(j) \tag{3.8}$$

*where*

$$s(j+1) = \begin{pmatrix}\begin{pmatrix}0 & E^T\end{pmatrix}\Gamma(j)s(j)\\ Hv_j\end{pmatrix}, \quad s(0) = 0, \tag{3.9}$$

$$\Gamma(j+1) = \begin{pmatrix} -\begin{pmatrix}0 & E^T\end{pmatrix}\Gamma(j)\begin{pmatrix}0\\E\end{pmatrix} & F^T\\ F & GG^T\end{pmatrix}^{-1}, \quad \Gamma(0) = 0. \tag{3.10}$$

The proof is obtained by straightforward application of the method of dynamic programming.

**Theorem 3.1** *The cost function $J(v)$ is given by*

$$J(v) = \sum_{j=1}^{N-1} s(j)^T\Gamma(j)s(j) + s(N)^T\Phi s(N) \tag{3.11}$$

*where the $s(j)$'s and $\Gamma(j)$'s are respectively defined in (3.9) and (3.10), and where*

$$\Phi = \Gamma(N) - \Gamma(N)\begin{pmatrix}0\\E\end{pmatrix}\left(\begin{pmatrix}0 & E^T\end{pmatrix}\Gamma(N)\begin{pmatrix}0\\E\end{pmatrix}\right)^{\dagger}\begin{pmatrix}0 & E^T\end{pmatrix}\Gamma(N). \tag{3.12}$$

**Proof** To compute $J(v)$, we can use (3.7) and (3.8) for $j = N$. It is straightforward to show that any $\xi^*(N)$ satisfying

$$\begin{pmatrix}0 & F^T\end{pmatrix}\Gamma(N)\begin{pmatrix}0\\F\end{pmatrix}\xi^*(N) = \begin{pmatrix}0 & F^T\end{pmatrix}\Gamma(N)s(N) \tag{3.13}$$

is an optimal solution of (3.7). Equation (3.11) is then obtained by placing

$$\xi^*(N) = \left(\begin{pmatrix}0 & F^T\end{pmatrix}\Gamma(N)\begin{pmatrix}0\\F\end{pmatrix}\right)^{\dagger}\begin{pmatrix}0 & F^T\end{pmatrix}\Gamma(N)s(N) \tag{3.14}$$

in (3.8) for $i = N$. ■

# 4 ASYMPTOTIC BEHAVIOR

## 4.1 The Algebraic Riccati Equation

To study the asymptotic behavior of the solution as $N$ goes to infinity, let

$$P(k) = \begin{pmatrix}0 & I\end{pmatrix}\Gamma(k)\begin{pmatrix}0\\I\end{pmatrix}. \tag{4.15}$$

Then, clearly

$$P(k) = \begin{pmatrix}0 & I\end{pmatrix}\begin{pmatrix}-E^TP(k-1)E & F^T\\ F & GG^T\end{pmatrix}^{-1}\begin{pmatrix}0\\I\end{pmatrix} \tag{4.16}$$

$$P(0) = 0. \tag{4.17}$$

**Theorem 4.1** *$P(k)$ converges exponentially fast to the unique positive definite solution of the algebraic descriptor Riccati equation*

$$P = \left(F\left(E^TPE\right)^{-1}F^T + GG^T\right)^{-1}. \tag{4.18}$$

**Proof** The proof has four parts. First we show that $P(k)$ is increasing. Then we show that it is upper-bounded. This proves that $P(k)$ converges. Then we show that the limit $P$ is positive definite. And finally we show that $P$ is the unique solution of the algebraic descriptor Riccati equation (4.18).

**Lemma 4.1** *The sequence of $P(k)$'s satisfies*

$$P(k+1) \geq P(k), \;\; \forall k \geq 0. \tag{4.19}$$

**Proof** First consider the optimization problem

$$\mathcal{V}(X,\phi) = \min_{\xi,\nu} \xi^TE^TXE\xi + \nu^T\nu \tag{4.20}$$

subject to $F\xi = \phi - G\nu$. Clearly,

$$X_1 \geq X_2 \implies \mathcal{V}(X_1,\phi) \geq \mathcal{V}(X_2,\phi), \; \forall\phi. \tag{4.21}$$

The solution to the optimization problem (4.20) is

$$\mathcal{V}(X,\phi) = \phi^T\begin{pmatrix}0 & I\end{pmatrix}\begin{pmatrix}-E^TXE & F^T\\ F & GG^T\end{pmatrix}^{-1}\begin{pmatrix}0\\I\end{pmatrix}\phi. \tag{4.22}$$

Thus by letting $X_1 = P(k-1)$ and $X_2 = P(k)$, we get

$$P(k) \geq P(k-1) \implies P(k+1) \geq P(k) \tag{4.23}$$

but $P(1) \geq 0 = P(0)$ so $P(k)$ is increasing and positive semi-definite. ■

**Lemma 4.2** *There exist a positive semi-definite matrix $\hat{P}$ such that*

$$P(k) \leq \hat{P}, \;\; \forall k \geq 0. \tag{4.24}$$

**Proof** From (2.4), it is easy to see that there exists an invertible matrix

$$Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \tag{4.25}$$

such that

$$\begin{pmatrix} E - zF & zG \end{pmatrix} \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} = \begin{pmatrix} zI + EY_1 & EY_2 \end{pmatrix} \tag{4.26}$$

where $\begin{pmatrix} Y_1 & Y_2 \end{pmatrix}$ is a right inverse of $\begin{pmatrix} -F & G \end{pmatrix}$. Clearly $\begin{pmatrix} zI + EY_1 & EY_2 \end{pmatrix}$ has full row rank, $\forall z \neq 0$, which implies that $(EY_1, EY_2)$, and consequently $(-EY_1, EY_2)$ is a stabilizable pair. Thus there exists a matrix $K$ such that $A = -EY_1 + EY_2 K$ is stable (has all of its eigenvalues inside the unit circle). Let

$$\begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = \begin{pmatrix} Y_1 + Y_2 K \\ Y_3 + Y_4 K \end{pmatrix} \tag{4.27}$$

Then

$$\begin{pmatrix} -F & G \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} = I \tag{4.28}$$

and $EL_1$ has all of its eigenvalues inside the unit circle. Now consider the following cost function

$$\hat{J}(z) = \sum_{k=0}^{N-1} \nu(k)^T \nu(k) \tag{4.29}$$

subject to

$$E\xi(k+1) = F\xi(k) + G\nu(k), \quad k = 0, \cdots, N-1, \tag{4.30}$$

$$z = \xi(N) \tag{4.31}$$

where we let

$$\nu(k) = L_2 E \xi(k+1). \tag{4.32}$$

This choice of $\nu$ yields

$$\xi(k) = -L_1 E \xi(k+1) \tag{4.33}$$

but the nonzero eigenvalues of $L_1 E$ are identical to those of $EL_1$ (which are inside the unit circle), thus recursion (4.33) is stable and $\xi(k)$ converges exponentially to zero. Then $\nu(k)$ also converges to zero thanks to (4.32). This implies that $\hat{J}(z)$ converges as $N$ goes to infinity, for all $z$, which implies that there exists a positive semi-definite matrix $\hat{Q}$ such that

$$\lim_{N \to \infty} \hat{J}(z) = z^T \hat{Q} z, \quad \forall z. \tag{4.34}$$

Now consider the same cost function but instead of the particular choice of $\nu$ used above, take the $\nu$ that minimizes the cost, i.e.,

$$J(\xi(N)) = \min \sum_{k=0}^{N-1} \nu(k)^T \nu(k) \tag{4.35}$$

subject to (4.30). This problem is of course the same as problem (3.5) with $v = 0$. The solution is

$$J(\xi(N)) = \xi(N)^T F^T P(N) F \xi(N), \quad \forall \xi(N). \tag{4.36}$$

Since the optimal solution is necessarily smaller than or equal to any particular solution, and thanks to the fact that the sequence $P(k)$ is increasing, we have that

$$E^T P(k) E \leq \hat{Q}, \quad \forall k, \tag{4.37}$$

which implies (4.24) where

$$\hat{P} = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} -\hat{Q} & F^T \\ F & GG^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{4.38}$$

■

So for we have shown that $P(k)$ is increasing and bounded, which implies that it converges to some $P$. Now we show that $P$ is positive definite. Suppose it is not and let $X$ be a matrix such that its columns form a basis for the null space of $P$, and let

$$\begin{pmatrix} S \\ T \end{pmatrix} = \begin{pmatrix} -E^T P E & F^T \\ F & GG^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ X \end{pmatrix}. \tag{4.39}$$

Note that

$$P = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} -E^T P E & F^T \\ F & GG^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{4.40}$$

and the image of $X$ is in the null space of $P$, so $T$ in (4.39) is zero. Thus, from (4.39) follows that

$$E^T P E S = 0 \tag{4.41}$$
$$F S = X \tag{4.42}$$

which implies that

$$P E S = 0 \tag{4.43}$$
$$P F S = 0 \tag{4.44}$$

and since $E$ and $F$ are full column rank, the columns of $ES$ and $FS$ form two bases for the null space of $P$. Thus there exists a square invertible matrix $L$ such that $ES = FSL$. Let $U$ be the matrix of change of basis that puts $L$ in Jordan form: $L = UJU^{-1}$ where $J$ is in Jordan form. Thus $ESU = FSUJ$, so if we denote the first column of $SU$ by $s$, we have

$$Es = J_{11} F s \tag{4.45}$$

where $J_{11}$ is the $(1,1)$ entry of $J$ (because $J$ is upper triangular[1]). $S$ has full column rank (because $X$ has

[1]Note that $s$ is an eigenvector of $\{E, F\}$, in fact each column of $SU$ is either an eigenvector or a generalized eigenvector of $\{E, F\}$.

full column rank), so $SU$ has full column rank which implies that $s$ is not zero. But then (4.45) contradicts Assumption (2.3). Thus $P$ is positive definite.

Finally, we must show that there is a unique positive definite solution to the algebraic descriptor Riccati equation. Suppose there are two distinct solutions $P_1$ and $P_2$, i.e.,

$$P_i = \left(F\left(E^T P_i E\right)^{-1} F^T + GG^T\right)^{-1}, \quad i = 1, 2. \tag{4.46}$$

By taking the inverse of both sides of (4.46) and subtracting the result for $i = 2$ from the result for $i = 1$, we get

$$\begin{aligned} P_1^{-1} - P_2^{-1} =& F(E^T P_1 E)^{-1} E^T P_1 (P_1^{-1} - P_2^{-1}) \\ & (F(E^T P_2 E)^{-1} E^T P_2)^T \end{aligned} \tag{4.47}$$

which implies that, for all $k \geq 1$,

$$\begin{aligned} P_1^{-1} - P_2^{-1} =& (F(E^T P_1 E)^{-1} E^T P_1)^k (P_1^{-1} - P_2^{-1}) \\ & ((F(E^T P_2 E)^{-1} E^T P_2)^T)^k. \end{aligned} \tag{4.48}$$

Clearly if we show that $F(E^T P_i E)^{-1} E^T P_i$, $i = 1, 2$, have all their eigenvalues inside the unit circle, we immediately have that $P_1 = P_2$. This can be shown by noting that thanks to (4.46),

$$\begin{aligned} P_i^{-1} - & (F(E^T P_i E)^{-1} E^T P_i) \\ & P_i^{-1} (F(E^T P_i E)^{-1} E^T P_i)^T = GG^T. \end{aligned} \tag{4.49}$$

But (4.49) is a Lyapunov equation and thus it is enough to show that

$$(F(E^T P_i E)^{-1} E^T P_i,\ G), \quad i = 1, 2,$$

are controllable pairs. Suppose this is not the case, i.e., there exists a $z$ and a non zero $w$ such that

$$w^T \begin{pmatrix} zI - F(E^T P_i E)^{-1} E^T P_i & G \end{pmatrix} = 0 \tag{4.50}$$

which implies that

$$\begin{aligned} w^T F(E^T P_i E)^{-1} E^T P_i &= z w^T & (4.51) \\ w^T G &= 0. & (4.52) \end{aligned}$$

Multiplying (4.51) on the right by $E$, and using (4.52), we obtain

$$w^T \begin{pmatrix} zE - F & G \end{pmatrix} = 0 \tag{4.53}$$

which is a contradiction (Assumption (2.4)). Thus both $F(E^T P_1 E)^{-1} E^T P_1$ and $F(E^T P_2 E)^{-1} E^T P_2$ have all their eigenvalues inside the unit circle. ■

The solution to the algebraic descriptor Riccati equation (4.18) can be constructed using the matrix pencil

$$\Psi = \left\{ \begin{pmatrix} F & GG^T \\ 0 & E^T \end{pmatrix}, \begin{pmatrix} E & 0 \\ 0 & F^T \end{pmatrix} \right\}. \tag{4.54}$$

**Theorem 4.2** *The matrix pencil $\Psi$ is regular, has no eigenmode on the unit circle and if the columns of $\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}$ form a basis for the stable eigenspace of $\Psi$, i.e.,*

$$\begin{pmatrix} F & GG^T \\ 0 & E^T \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \mathcal{J} = \begin{pmatrix} E & 0 \\ 0 & F^T \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \tag{4.55}$$

*where eigenvalues of $\mathcal{J}$ are inside the unit circle, then*

$$P = (F\Gamma_1\Gamma_2^{-1} + GG^T)^{-1} \tag{4.56}$$

*is the unique positive definite solution of the algebraic descriptor Riccati equation (4.18).*

**Proof** Let

$$\begin{aligned} \Psi(z) &= z \begin{pmatrix} F & GG^T \\ 0 & E^T \end{pmatrix} - \begin{pmatrix} E & 0 \\ 0 & F^T \end{pmatrix} \\ &= \begin{pmatrix} zF - E & zGG^T \\ 0 & zE^T - F^T \end{pmatrix}. \end{aligned} \tag{4.57}$$

Suppose $z$ is on the unit circle and let $z^*$ denote the complex conjugate of $z$. Note that $z^* = 1/z$. To show that $z$ is not an eigenmode of $\Psi$, we must show that $\Psi(z)$ is invertible, or equivalently that

$$\begin{pmatrix} zF - E & GG^T \\ 0 & z^* F^T - E^T \end{pmatrix}$$

is invertible. Suppose this is not the case, i.e., there exist $x$ and $y$, not both zero, such that

$$\begin{pmatrix} zF - E & GG^T \\ 0 & z^* F^T - E^T \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{4.58}$$

But this implies, after premultiplication of the first equation by the complex conjugate transpose of $y$, that

$$\begin{pmatrix} G^T \\ z^* F^T - E^T \end{pmatrix} y = 0. \tag{4.59}$$

Thanks to Assumption (2.4), (4.59) implies that $y = 0$, which in turn implies that

$$\left(zF - E\right) x = 0 \tag{4.60}$$

which implies that $x = 0$, thank to Assumption (2.3). But this is a contradiction, so $\Psi$ has no eigenmode on the unit circle.

Let $p(z)$ denote the determinant of $\Psi(z)$, and $\mu$ the degree of $p(z)$. Thanks to the identity

$$\left[\begin{pmatrix} 0 & -I \\ z^{-1}I & 0 \end{pmatrix} \Psi(z)\right]^T = \begin{pmatrix} 0 & -I \\ zI & 0 \end{pmatrix} \Psi(z^{-1}), \tag{4.61}$$

by taking the determinant of both sides, we get $z^{-m}p(z) = z^m p(z^{-1})$ where $m$ equals the number of rows of $E$. So, since $p$ does not have any roots on the unit circle, $\mu = m$ and consequently $\Gamma_2$ is square.

From (4.55), we get

$$\begin{aligned} F\Gamma_1\mathcal{J} + GG^T\Gamma_2\mathcal{J} &= E\Gamma_1 \qquad (4.62)\\ E^T\Gamma_2\mathcal{J} &= F^T\Gamma_2 \qquad (4.63)\end{aligned}$$

which implies that

$$\Gamma_2^T F\Gamma_1 = \mathcal{J}^T\Gamma_2^T F\Gamma_1\mathcal{J} + \mathcal{J}^T\Gamma_2^T GG^T\Gamma_2\mathcal{J} \qquad (4.64)$$

which is a Lyapunov equation and since $\mathcal{J}$ has all its eigenvalues inside the unit circle,

$$W = \Gamma_2^T F\Gamma_1 \qquad (4.65)$$

is symmetric positive semi-definite.

**Lemma 4.3** *The matrix $\Gamma_2$ is invertible.*

**Proof** Suppose $\Gamma_2 w = 0$ which implies that $Ww = 0$. Thanks to (4.64), we get that

$$\begin{aligned} W\mathcal{J}w &= 0 \qquad (4.66)\\ G^T\Gamma_2\mathcal{J}w &= 0. \qquad (4.67)\end{aligned}$$

But from (4.63) we get $E^T\Gamma_2\mathcal{J}w = 0$ which thanks to (4.67) and (2.4) implies that $\Gamma_2\mathcal{J}w = 0$. Thus $\ker(\Gamma_2)$ is $\mathcal{J}$-invariant. This implies that there exists at least one eigenvector of $\mathcal{J}$ in $\ker(\Gamma_2)$, i.e., there exist a non-zero vector $v$ and a scalar $\lambda$ such that $\Gamma_2 v = 0$ and $\mathcal{J}v = \lambda v$. So by multiplying (4.62) on the right by $v$ we obtain

$$(\lambda F - E)\Gamma_1 v = 0, \qquad (4.68)$$

which thanks to (2.4) and (2.4) implies that $\Gamma_1 v = 0$. But this is a contradiction because $\begin{pmatrix}\Gamma_1\\ \Gamma_2\end{pmatrix}$ has full column rank. Thus $\Gamma_2$ is invertible. ∎

**Lemma 4.4** *The following always holds*

$$\ker(F^T\Gamma_2) = \ker(\Gamma_1). \qquad (4.69)$$

**Proof** Since $F$ has full column rank and $\Gamma_2$ is invertible, $\ker(W) = \ker(\Gamma_1)$ and since $W$ is symmetric

$$\ker(F^T\Gamma_2) \subset \ker(\Gamma_1). \qquad (4.70)$$

Now we show that

$$\ker(\mathcal{J}) = \ker(F^T\Gamma_2). \qquad (4.71)$$

From (4.63) it follows that $\ker(\mathcal{J}) \subset \ker(F^T\Gamma_2)$. Let $w$ be any vector such that $F^T\Gamma_2 w = 0$, this implies, thanks to (4.64), that

$$G^T\Gamma_2\mathcal{J}w = 0 \qquad (4.72)$$

and thanks to (4.63), that

$$E^T\Gamma_2\mathcal{J}w = 0. \qquad (4.73)$$

But (4.72) and (4.73), because of Assumption (2.4), imply that $\Gamma_2\mathcal{J}w$ and consequently $\mathcal{J}w$ is zero. Thus $\ker(F^T\Gamma_2) \subset \ker(\mathcal{J})$. This proves (4.71).

Now we show that

$$\ker(\mathcal{J}) = \ker(\Gamma_1). \qquad (4.74)$$

From (4.62) and the full rankedness of $E$, it is easy to see that $\ker(\mathcal{J}) \subset \ker(\Gamma_1)$.

Let $w$ be any vector satisfying $\Gamma_1 w = 0$, then by premultiplying by $w^T$ and postmultiplying by $w$ (4.64), we obtain

$$\begin{aligned} \Gamma_1\mathcal{J}w &= 0 \qquad (4.75)\\ G^T\Gamma_2\mathcal{J}w &= 0. \qquad (4.76)\end{aligned}$$

From (4.75) we get that $\ker(\Gamma_1)$ is $\mathcal{J}$-invariant. Let $v$ be any eigenvector of $\mathcal{J}$ satisfying $\Gamma_1 v = 0$, and $\lambda$ the associated eigenvalue, i.e., $\mathcal{J}v = \lambda v$. Then,

$$\begin{aligned} \lambda E^T\Gamma_2 v &= F^T\Gamma_2 v = 0 \qquad (4.77)\\ \lambda G^T\Gamma_2 v &= 0. \qquad (4.78)\end{aligned}$$

Thus $\lambda$ is necessarily zero. So the restriction of $\mathcal{J}$ to $\ker\Gamma_1$ is nilpotent. We denote it by $\mathcal{N}$.

Now suppose $\ker(\Gamma_1)$ is not a subset of $\ker(\mathcal{J})$, i.e., there exists a vector $w$ such that $\Gamma_1 w = 0$ but $\mathcal{J}w \neq 0$. This clearly implies that $\mathcal{N} \neq 0$, i.e., the nilpotent matrix $\mathcal{N}$ has non trivial Jordan blocks which in turn implies that there exists a vector $v \in \ker(\Gamma_1)$ such that

$$\begin{aligned} \mathcal{J}v &\neq 0 \qquad (4.79)\\ \mathcal{J}^2 v &= 0. \qquad (4.80)\end{aligned}$$

But then from (4.64) and (4.62) follows that

$$\begin{aligned} F^T\mathcal{J}v &= 0 \qquad (4.81)\\ G^T\mathcal{J}v &= 0 \qquad (4.82)\end{aligned}$$

which because of Assumption (2.4), imply that $\mathcal{J}v = 0$. But this is a contradiction. This shows (4.74). Finally, (4.69) follows from (4.71) and (4.74). ∎

Now we can show that

$$\mathcal{H} = F\Gamma_1\Gamma_2^{-1} + GG^T \qquad (4.83)$$

is invertible. Note that $F\Gamma_1\Gamma_2^{-1} = (\Gamma_2^{-1})^T W\Gamma_2^{-1}$ is positive semi-definite. So if $\mathcal{H}w = 0$, then

$$\begin{aligned} W\Gamma_2^{-1}w &= 0 \qquad (4.84)\\ G^T w &= 0. \qquad (4.85)\end{aligned}$$

But thanks to (4.69) and full rankedness of $F$, $\ker(W) = \ker(F^T\Gamma_2)$. Thus (4.84) and (4.85) imply that $w^T \begin{pmatrix} F & G \end{pmatrix} = 0$, which implies that $w = 0$. Thus $\mathcal{H}$ is invertible and positive-definite. From (4.62) it follows that $\Gamma_2 \mathcal{J} = \mathcal{H}^{-1} E \Gamma_1$ and from (4.63), that

$$E^T \mathcal{H}^{-1} E \Gamma_1 \Gamma_2^{-1} = F^T. \qquad (4.86)$$

But $E$ has full column rank, so that $E^T \mathcal{H}^{-1} E$ is invertible. Thus from (4.86) we obtain $F \Gamma_1 \Gamma_2^{-1} = F(E^T \mathcal{H}^{-1} E)^{-1} F^T$. But then thanks to (4.83), we obtain

$$\mathcal{H} = GG^T + F(E^T \mathcal{H}^{-1} E)^{-1} F^T. \qquad (4.87)$$

By letting

$$P = \mathcal{H}^{-1}, \qquad (4.88)$$

we obtain the algebraic descriptor Riccati equation (4.18). Noting that (4.88) is equivalent to (4.56) Theorem 4.2 is proved. ∎

## 5 CONCLUSION

We have proposed a recursive solution to a very general discrete LQ optimization problem. We have studied the asymptotic properties of this problem and given a constructive solution for the limiting case as the length of the interval goes to infinity.

## REFERENCES


[1] Bellman, R. E. and S. E. Dreyfus (1962). *Applied dynamic programming*. Princeton University Press.

[2] Bernhard, P. and X. M. Wang (1987). Filtrage des systèmes implicites linéaires discrètes, *C. R. Acad. Sc. Paris*, **t. 304, Série I, n$^o$ 12**, 351-355.

[3] Brenan, K. E. , S. L. Campbell and L. R. Petzold (1996). *The Numerical Solution of Initial Value Problems in Differential-Algebraic Equations*. SIAM.

[4] Dai, L. (1989). Filtering and LQG problems for discrete-time stochastic singular systems. *IEEE Trans. on Automat. Control*, **AC-34**, 1105-1108.

[5] Campbell, S. L. and C.D. Meyer, Jr. (1991). *Generalized Inverses of Linear Transformations*. Dover, New York.

[6] Darouach, M., M. Zasadzinski, A. Basson Onana and S. Nowakowski (1995). Kalman filtering with unknown inputs via optimal state estimation of singular systems. *Int. J. Systems Sci.*, **Vol. 26, No. 10**, 2015-2028.

[7] Darouach, M., M. Zasadzinski and D. Mehdi (1993). State estimation of stochastic singular linear systems. *Int. J. Systems Sci.*, **Vol. 2, No. 2**, 345-354.

[8] Levy, B. C., A. Benveniste and R. Nikoukhah (1996), High-level primitives for recursive maximum likelihood estimation, *IEEE Trans. on Automat. Control*, **AC-41**, 1125-1145.

[9] Nikoukhah, R., S. L. Campbell, F. Delebecque, Observer design for general linear time-invariant systems, *Automatica*, **Vol. 34**, 575-583.

[10] Nikoukhah, R., S. L. Campbell, F. Delebecque, Kalman filtering for general discrete-time linear systems (1999), *IEEE Trans. Automat. Control*, **AC-44**, 1829-1839. **Vol. 34**, 575-583.

[11] Nikoukhah, R., D. Taylor, R., A. S. Willsky and B. C. Levy (1995). Graph Structure and Recursive Estimation of Noisy Linear Relations. *J. of Math. Systems, Estimation, and Control*, **5(4)**, 1-37.

[12] Nikoukhah, R., A. S. Willsky and B. C. Levy (1992). Kalman filtering and Riccati equations for descriptor systems. *IEEE Trans. Automat. Control*, **AC-37**, 1325-1342.