Auxiliary signal design for failure detection in uncertain systems

R. Nikoukhah, S. L. Campbell and F. Delebecque

Abstract—**An auxiliary signal is an input signal that enhances the identifiability of a model based on input-output observations. Assuming that the normal and the failed behaviors of a process can be modeled by two linear uncertain systems, failure detectability can be seen as a multi-model identification problem. In this paper, we extend previous results on auxiliary signal design for multi-model identification to a larger class of uncertain systems.**

Keywords—**Identification, Failure detection, Auxiliary signal design**

I. Introduction

There are two approaches to the problem of failure detection and isolation. The first is a passive approach where the detector monitors the inputs and the outputs of the system and decides whether (and if possible what kind of) a failure has occurred. This is done by comparing the measured input-output behavior with "normal" behavior of the system. The passive approach is used to continuously monitor the system in particular when the detector has no way of acting upon the system, for material or security reasons. Most of the work in the area of failure detection is geared towards this type of approach [1], [12], [15].

The active approach to failure detection consists in acting upon the system on a periodic basis or at critical times using a test signal, which we call an auxiliary signal, in order to exhibit abnormal behaviors which would otherwise remain undetected during normal operation. The detector can act by taking over part or all of the inputs of the system for a period of time: the test period. The decision whether or not the system has failed should be made at (and if possible before) the end of the test period. The structure of the failure detection method considered in this paper is depicted in Figure 1.

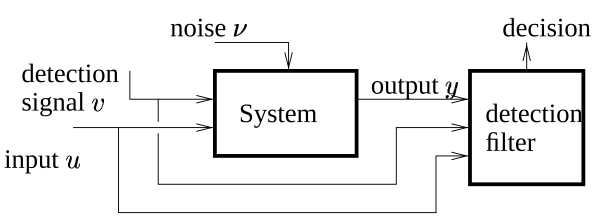

Fig. 1. Active failure detection.

The design of auxiliary signals has been a major issue in system identification but their use for failure detection has been introduced in [16], [6], [7]; see also [14]. The auxiliary signal in these works are considered to be linear inputs of stochastic models and their objective is to optimize some statistical properties of the detector. In [9], a method for guaranteed failure detection was presented in which perturbations were modeled as polyhedral sets. The method was based on solving large linear programming problems.

The auxiliary signal design problem for robust multi-model identification has been studied [10], [11]. In [10], [11], we assume that we have two candidate models and we seek an auxiliary signal of least energy that can guarantee on-line identification of the correct model.

In this paper, we consider the same problem but we allow for more general noise models. This enables us, in particular, to capture model uncertainties more effectively.

II. Problem formulation

A. System models

The true system model is supposed to be one of the following two models over the test period $[0, T]$:

$$\dot{x}_i = A_i x_i + B_i v + M_i \nu_i, \quad (1)$$

$$z_i = G_i x_i + H_i v, \quad (2)$$

$$y = C_i x_i + N_i \nu_i \quad (3)$$

for $i = 0$ and 1. v is the auxiliary signal input, y is the measured output, x_i's are the states and ν_i's are uncertainty inputs and z_i's represent uncertainty outputs. A_i, B_i, C_i, G_i, H_i, M_i, N_i are matrices of appropriate dimensions. These matrices could be time varying but each entry must be a continuous function of time. Finally, we assume that the N_i's have full row rank.

The assumption on the uncertainty inputs and outputs are

$$\mathcal{S}_v^i(x_i(0), \nu_i) \triangleq x_i(0)^T P_{i,0}^{-1} x_i(0) + \int_0^T (\|\nu_i\|^2 - \alpha_i^2 \|z_i\|^2) dt < 1, \quad i = 0, 1. \quad (4)$$

$\mathcal{S}_v^i(x_i(0), \nu_i)$ measures the size of the disturbances that our model identification is to be robust to.

Note that the states of the two models x_0 and x_1 need not have the same dimensions. The same is true for uncertainties(ν_0, z_0) and (ν_1, z_1). The only things that tie together the two models are y and v.

This formulation of uncertain systems allows us to model uncertain systems represented as follows:

$$\dot{x}_i = (A_i + M_i \Delta_i G_i) x_i + (B_i + M_i \Delta_i H_i) v, \quad (5)$$

$$y = (C_i + N_i \Delta_i G_i) x_i + N_i \Delta_i H_i v \quad (6)$$

where

$$\|\Delta_i\| < \alpha_i, \quad i = 0, 1, \quad (7)$$

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and where the initial condition satisfies:

$$x_i(0)^T P_{i,0}^{-1} x_i(0) < 1, \quad i = 0, 1. \tag{8}$$

The set of uncertain systems modeled this way is included in the set of uncertain systems modeled by (1)-(4). To see this, simply let

$$\nu_i = \Delta_i z_i, \quad i = 0, 1. \tag{9}$$

See [13] for more details.

We shall make the assumption hereafter that $\mathcal{S}_v^i(x_i(0), \nu_i)$, $i = 0, 1$, are convex functions. This condition which is slightly stronger that "strict verifiability" in [13] is always true for sufficiently small α_i's. This assumption can be somewhat relaxed as we shall see later.

B. Proper auxiliary signal

We say that the $\mathcal{L}^2[0, T]$ vector function v is a *proper auxiliary signal* if its application implies that we are able to always distinguish the two candidate models based on observation y.

Definition II.1: The auxiliary signal v is not proper if there exist x_0, x_1, ν_0, ν_1, z_0, z_1 and y satisfying (1), (2), (3) and (4) both for $i = 0$ and $i = 1$. The auxiliary signal v is called proper otherwise.

Definition II.2: Let V denote the set of proper auxiliary signals v. Then,

$$\gamma^* = \left(\inf_{v \in V} \int_0^T ||v||^2 dt \right)^{-\frac{1}{2}}, \tag{10}$$

is called the separability index associated with (1)-(2).

Clearly, γ^{*-2} is a lower bound on the energy of proper auxiliary signals. So, greater separability index implies existence of lower energy proper auxiliary signal. The separability index is zero when there is no proper auxiliary signal. Note that if a proper auxiliary signal exists, then the minimum norm is nonzero since the N_i are surjective.

Auxiliary signal v being not proper means that there exist x_0, ν_0, x_1, ν_1, z_0, z_1 and y compatible with (1)-(2), $i = 0, 1$, such that

$$\max \left\{ \mathcal{S}_v^0(x_0(0), \nu_0), \mathcal{S}_v^1(x_1(0), \nu_1) \right\} < 1. \tag{11}$$

We can rewrite this inequality as follows

$$\max_{0 \le \beta \le 1} L_v((x_0, \nu_0, x_1, \nu_1), \beta) < 1 \tag{12}$$

where

$$L_v((x_0, \nu_0, x_1, \nu_1), \beta) = \beta \mathcal{S}_v^0(x_0(0), \nu_0) + (1 - \beta) \mathcal{S}_v^1(x_1(0), \nu_1). \tag{13}$$

We thus obtain the following characterization of properness.

Lemma II.1: The auxiliary signal v is not proper if and only if

$$\inf \max_{0 \le \beta \le 1} L_v((x_0, \nu_0, x_1, \nu_1), \beta) < 1 \tag{14}$$

where the infimum is taken over x_i, ν_i and y in $\mathcal{L}^2[0, T]$, subject to (1)-(3), $i = 0, 1$.

Note that the output y can easily be eliminated from the constraints (1)-(3), $i = 0, 1$, by subtracting (3) for $i = 0$ from (3) for $i = 1$, giving

$$0 = C_0 x_0 - C_1 x_1 + N_0 \nu_0 - N_1 \nu_1. \tag{15}$$

We thus end up with an infimum taken over S_v, the set of $\mathcal{L}^2[0, T]$ functions (x_0, ν_0, x_1, ν_1) satisfying the constraints (1), $i = 0, 1$, and (15).

Let us now define the following function which will prove very useful later on,

$$J_v(\beta) = \inf_{(x_0, \nu_0, x_1, \nu_1) \in S_v} L_v((x_0, \nu_0, x_1, \nu_1), \beta). \tag{16}$$

Lemma II.2: For all $v \in \mathcal{L}^2[0, T]$, for $0 \le \beta \le 1$, $J_v(\beta)$ is defined and has the following properties:

1. it is zero for $\beta = 0$ and $\beta = 1$,
2. it is quadratic in v, i.e., for all scalar c, $J_{cv}(\beta) = |c|^2 J_v(\beta)$.
3. it is a continuous and concave function of β,

Lemma II.3: For all $v \in \mathcal{L}^2[0, T]$, the infimum is attained in (16), and if $0 < \beta < 1$, the minimum is unique and continuous in β.

Theorem II.1: The function L has at least one saddle point $((x_0^s, \nu_0^s, x_1^s, \nu_1^s), \beta^s)$ on $S_v \times [0, 1]$ and

$$\begin{aligned} &\inf_{(x_0, \nu_0, x_1, \nu_1) \in S_v} \max_{\beta \in [0,1]} L_v((x_0, \nu_0, x_1, \nu_1), \beta) = \\ &\min_{(x_0, \nu_0, x_1, \nu_1) \in S_v} \max_{\beta \in [0,1]} L_v((x_0, \nu_0, x_1, \nu_1), \beta) = \\ &\max_{\beta \in [0,1]} \min_{(x_0, \nu_0, x_1, \nu_1) \in S_v} L_v((x_0, \nu_0, x_1, \nu_1), \beta) = \\ &L_v((x_0^s, \nu_0^s, x_1^s, \nu_1^s), \beta^s). \end{aligned} \tag{17}$$

Corollary II.1: Let $((x_0^s, \nu_0^s, x_1^s, \nu_1^s), \beta^s)$ be a saddle point of L on $S_v \times [0, 1]$. If $\beta^s \ne 0$ and $\beta^s \ne 1$, then

$$\mathcal{S}_v^0(x_0^s(0), \nu_0^s) = \mathcal{S}_v^1(x_1^s(0), \nu_1^s). \tag{18}$$

Our original problem which consisted in finding a minimum energy proper auxiliary signal v can now be expressed as follows:

$$\inf ||v|| \text{ subject to } \max_{0 < \beta < 1} J_v(\beta) \ge 1. \tag{19}$$

Note that we have excluded the cases $\beta = 0$ and $\beta = 1$ because $J_v(0) = J_v(1) = 0$ as shown in Lemma II.2.

Using the fact that $J_v(\beta)$ is quadratic in v (Lemma II.2), we obtain the following fundamental result:

Theorem II.2: Let

$$J^*(\beta) = \sup_{v \ne 0} \frac{J_v(\beta)}{\int_0^T ||v||^2 dt}. \tag{20}$$

Then

$$\gamma^{*2} = \max_{0 < \beta < 1} J^*(\beta) \tag{21}$$

where γ^* is the separability index defined previously.

Note that the larger γ^* is, the easier it is to separate the two models. And when $\gamma^* = 0$, then the two models are indistinguishable no matter what the input v is. So, γ^* can be considered as the deterministic counterpart of the Kullback distance [8] between two systems used in some stochastic formulations. Another interpretation of γ^* is in terms of *signal to noise ratio*: $1/\gamma^*$ can be seen as the smallest signal to noise ratio required for perfect identification.

III. Computation of the separability index

The separability index can be computed if we can compute $J^*(\beta)$.

To simplify the notations, let

$$x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \ \nu = \begin{pmatrix} \nu_0 \\ \nu_1 \end{pmatrix}, \ z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix},$$
$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \ M = \begin{pmatrix} M_0 & 0 \\ 0 & M_1 \end{pmatrix}, \ N = \begin{pmatrix} N_0 & -N_1 \end{pmatrix},$$
$$C = \begin{pmatrix} C_0 & -C_1 \end{pmatrix}, B = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}, G = \begin{pmatrix} G_0 & 0 \\ 0 & G_1 \end{pmatrix}, H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}$$
$$P_\beta^{-1} = \begin{pmatrix} \beta P_{0,0}^{-1} & 0 \\ 0 & (1-\beta)P_{1,0}^{-1} \end{pmatrix}, \ V_\beta = \begin{pmatrix} \beta I & 0 \\ 0 & (1-\beta)I \end{pmatrix},$$
$$W_\beta = \begin{pmatrix} \alpha_0 \beta I & 0 \\ 0 & \alpha_1(1-\beta)I \end{pmatrix}. \quad (22)$$

Then the constraints (1), $i = 0, 1$, and (15) can be expressed as

$$\dot{x} = Ax + Bv + M\nu \quad (23)$$
$$z = Gx + Hv \quad (24)$$
$$0 = Cx + N\nu \quad (25)$$

and the function $J_v(\beta)$ becomes

$$J_v(\beta) = \min_{(x,\nu)\in S_v} L_v((x,\nu),\beta) \quad (26)$$

where (with slight abuse of notation)

$$S_v = \{(x,\nu) \in \mathcal{L}^2[0,T] \times \mathcal{L}^2[0,T] \mid \text{ (23) and (25) hold}\} \quad (27)$$

and

$$L_v((x,\nu),\beta) = x(0)^T P_\beta^{-1} x(0) + \int_0^T \nu^T V_\beta \nu - z^T W_\beta z \, dt. \quad (28)$$

All the results presented so far hold as long as System (23)-(25) with

$$L_v((x,\nu),\beta) < 1 \quad (29)$$

is strictly verifiable. Strict verifiability here means that the set of (x,ν) satisfying (23)-(25) and (29) is bounded, or equivalently that $L_v((x,\nu),\beta)$ is convex in (x,ν) on $\mathcal{S}_v$. This condition is of course implied by our original assumptions.

Lemma III.1: System (23)-(25) with noise model (29) is strictly verifiable if and only if the Riccati equation

$$\dot{P} = AP + PA^T - P(C^T R_\beta^{-1} C - G^T W_\beta G)P + Q_\beta - S_\beta R_\beta^{-1} S_\beta^T, \ \ P(0) = P_\beta \quad (30)$$

has a positive-definite solution P on $[0,T]$.
This result is a straightforward extension of Theorem 4.3.1 in [13].

Note that as α_i's go to zero (and thus W_β goes to zero), the Riccati equation (30) converges to the standard Kalman filtering Riccati equation which always has a solution.

A. Computation of $J^*(\beta)$

Since $J_v(\beta)$ is a quadratic function of β, we can transform the optimization problem (20) as follows:

$$\max J_v(\beta) - \gamma^2 \int_0^T \|v\|^2 \, dt \quad (31)$$

where $J^*(\beta)$ is obtained as the smallest value of γ^2 for which the solution is not unbounded.

Theorem III.1: $\gamma^2 > J^*(\beta)$ if and only if the Riccati equation

$$\begin{aligned}\dot{\Pi} = {} & (A - S_\beta R_\beta^{-1} C - B(\gamma^2 I + H^T W_\beta H)^{-1} H^T W G)\Pi + \\ & \Pi(A - S_\beta R_\beta^{-1} C - B(\gamma^2 I + H^T W_\beta H)^{-1} H^T W G)^T \\ & - \Pi(C^T R_\beta^{-1} C - G^T (W_\beta^{-1} + \gamma^{-2} H H^T)^{-1} G)\Pi + \\ & Q_\beta - S_\beta R_\beta^{-1} S_\beta^T - B(\gamma^2 I + H^T W_\beta H)^{-1} B^T, \ \ \Pi(0) = P_\beta \end{aligned} \quad (32)$$

has a solution on $[0,T]$ where

$$\begin{pmatrix} Q_\beta & S_\beta \\ S_\beta^T & R_\beta \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} V_\beta^{-1} \begin{pmatrix} M \\ N \end{pmatrix}^T. \quad (33)$$

Since the value of $J^*(\beta)$ is determined by the existence of a solution to the Riccati equation (32), it is easy to see that there is a relationship between $J^*(\beta)$ and the interval length T. In particular, the larger T is, the larger $J^*(\beta)$ is going to be.

The algorithm for determining $J^*(\beta)$ works as follows. Pick a γ, solve the Riccati equation (32) with a standard ode (ordinary differential equation) solver and see how far it goes. If it goes beyond T, then reduce γ and start over. If the solution of the Riccati equation diverges before, increase γ and start over. Using a simple bisection method, $J^*(\beta)$ can be found with desired accuracy.

Note that for $\gamma = \infty$, this Riccati equation reduces to (30) which, we know, has a solution.

Note however that for $\gamma = J^*(\beta)$, the Riccati equation (32) may diverge at $t^* < T$. In most cases, this does not happen. For example, it can be shown that if the models are time-invariant, the Riccati equation always diverges at $t = T$. To construct an academic example where $t^* < T$, consider the case where the matrix C is identically zero over the interval $[t^*, T]$ for some $t^* < T$. If the Riccati equation (32) has a solution over $[0, t^*]$, then it has a solution over $[0,T]$ because this Riccati equation on $[t^*, T]$ is just a linear equation. So if γ is such that (32) doesn't have a solution on $[0, T]$, it cannot have a solution on $[0, t^*]$ either.

To see the reason for this behavior, note that in this case, the observation of y over $[t^*, T]$ does not provide any useful information for identification purposes. The separability index which is an increasing function of the length of the test period, is constant from t^* on. In this case, there is no point using a test period with $T > t^*$ since we are not going to have a superior separability index anyway.

So, in the unlikely event that the Riccati equation (32) diverges at $t^* < T$, we simply reduce the test period by letting $T = t^*$. We will have then a faster identification scheme without having to pay a higher price in terms of the auxiliary signal energy. In the sequel, we shall suppose that the Riccati equation (32) diverges at $t = T$ and has a solution on $[0, T)$.

B. *Computation of the separability index γ^**

In the previous section, we have seen how to compute $J^*(\beta)$, for any given β. Here, we use this to compute the separability index, based on (21). Even though $J^*(\beta)$ is not concave, it has nice properties making the optimization problem (21) numerically tractable. In particular, thanks to Lemma II.2, we see that $J^*(\beta)$ is the supremum over concave functions each of which is zero at $\beta = 0$ and $\beta = 1$. Using this fact, we obtain the following result.

Lemma III.2: For any β_1 and β_2 satisfying $0 \le \beta_1 < \beta_2 \le 1$,

$$\max_{\beta_1 \le \beta \le \beta_2} J^*(\beta) \le \frac{J^*(\beta_1) J^*(\beta_2)}{J^*(\beta_1)(1-\beta_2) + J^*(\beta_2)\beta_1}. \tag{34}$$

The proof follows a straightforward geometric argument illustrated in Figure 2. The key idea is that $J^*(\beta)$ is the supremum over concave functions going through the points $(0, 0)$ and $(1, 0)$, so it remains necessarily below the two dashed lines inside $[\beta_1, \beta_2]$ (and above outside).

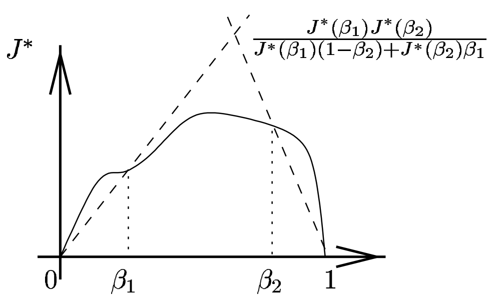

Fig. 2. A typical $J^*(\beta)$ as a function of β.

Now consider the following simple optimization strategy for estimating γ^{*2} which consists of taking the maximum of $J^*(\beta)$ for n regularly spaced values of β over $[0, 1]$:

$$\hat{\gamma}^2 = \max_{k=1,..,n} J^*(k/(n+1)). \tag{35}$$

Then, thanks to Lemma III.2, it is straightforward to show that

$$\left| \frac{\gamma^{*2} - \hat{\gamma}^2}{\hat{\gamma}^2} \right| \le \frac{1}{n+1}. \tag{36}$$

So for example by simply taking $\beta = 1/2$, the worst possible error in γ^{*2} would be a factor of two.

The optimization method (35), thanks to (36), allows us to compute γ^* with the desired precision. We can also use more sophisticated algorithms to estimate γ^* and even couple the β and γ iterations.

IV. Construction of the auxiliary signal

Once we have computed β^*, an optimal value of β in (21), and the separability index γ^*, we can proceed with the construction of minimal energy proper auxiliary signal v.

Theorem IV.1: An optimal auxiliary signal v is given as any non-trivial solution of the two-point boundary-value system:

$$\frac{d}{dt}\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}\begin{pmatrix} x \\ \lambda \end{pmatrix} \tag{37}$$

with boundary conditions:

$$x(0) = P_{\beta^*}\lambda(0) \tag{38}$$

$$\lambda(T) = 0. \tag{39}$$

where

$$\Omega_{11} = -\Omega_{22}^T = A - S_{\beta^*}R_{\beta^*}^{-1}C - B\Lambda^{-1}H^T W_\beta G \tag{40}$$

$$\Omega_{12} = Q_{\beta^*} - S_{\beta^*}R_{\beta^*}^{-1}S_{\beta^*}^T - B\Lambda^{-1}B^T \tag{41}$$

$$\Omega_{21} = C^T R_{\beta^*}^{-1}C - G^T(W_\beta^{-1} + \gamma^{*-2}HH^T)^{-1}G \tag{42}$$

and where

$$\Lambda = \gamma^{*2}I + H^T W_\beta H. \tag{43}$$

We already know that this two-point boundary-value system has non-zero solutions for β^* and γ^* that we have computed. It is exactly these solutions which give us the optimal auxiliary signal as

$$v^* = \frac{\Lambda^{-1}(-H^T W_\beta G x + B^T\lambda)}{\gamma^* \sqrt{\int_0^T \|\Lambda^{-1}(-H^T W_\beta G x + B^T \lambda)\|^2 \, dt}}. \tag{44}$$

Note that the norm of v^* equals $1/\gamma^*$ giving $J_{v^*}(\beta^*) = 1$ which is exactly the limit that guarantees that v^* is proper (see (19)).

V. Conclusions

We have presented a methodology for error-free system identification in the situation where we have two candidate uncertain linear models and where we have control over the input. A method for the construction of an optimal input (auxiliary signal) is given. This work has applications to failure detection and identification.

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