ACTIVE FAILURE DETECTION: AUXILIARY SIGNAL DESIGN AND ON-LINE DETECTION

R. Nikoukhah*, **S. L. Campbell**†

*INRIA, Rocquencourt BP 105, 78153 Le Chesnay Cedex, France.
fax: +(33) 1-39-63-57-86
e-mail: ramine.nikoukhah@inria.fr

†Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA.
fax: 1-919-515-3798 e-mail: slc@math.ncsu.edu
http://www.math.ncsu.edu/~slc

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Abstract

This paper describes an active approach for model identification and failure detection in the presence of quadratically bounded uncertainty. After developing the underlying geometry, two particular examples of this approach involving static and continuous models are described. Several examples are given.

1 Introduction

There are two general approaches to failure detection and isolation. One is a passive approach where a detector monitors input and outputs of the system and decides whether, and if possible what kind of, a failure has occurred. A passive approach is used for continuous monitoring. The detector has no way of acting upon the system.

In contrast, in an active approach the system is acted upon on a periodic basis or at critical times using a test signal (auxiliary signal) to exhibit abnormal behaviors. The decision of whether or not the system has failed should be made by the end of the test period. The active approach has the advantages that it can sometimes detect failures that are not detectable during the normal operation of the system. This is especially important for evaluating subsystem status before the subsystem's performance becomes crucial. An example would be evaluating the brakes while moving but before a truck has to stop. An active approach also often permits quicker detection of a failure. Of course, it is usually important that the test signal be small in some sense in order to not interfere with normal operation.

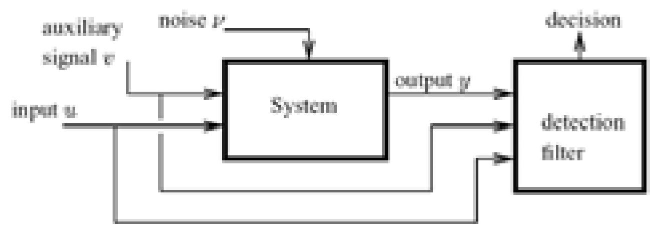

Figure 1: General system structure.

In [2, 3, 4, 5] we have begun the investigation of a multi-model active approach for model identification and failure detection. These earlier papers have focused on the theory and computation for various special cases. As this work has progressed a general framework encompassing all of these cases and several additional ones has begun to become evident. This paper will discuss this general framework for the first time.

2 Geometry of the approach

The structure of the active failure detection method considered here is described in Figure 1. The system is acted on by both a control input u and the auxiliary signal v. The system is subject to noise ν and there is an output y. The information available for

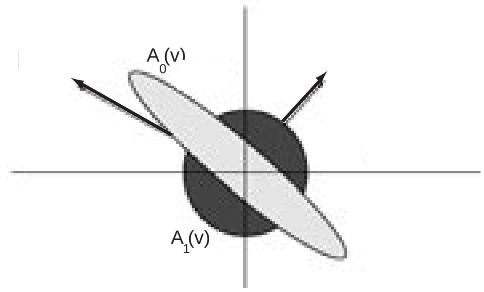

Figure 2: No guaranteed detection.

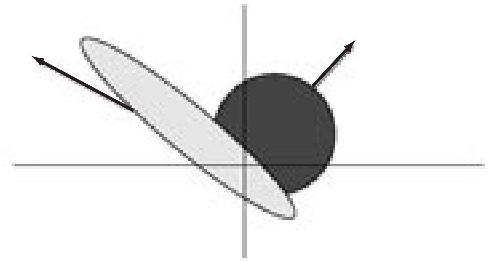

Figure 3: Increasing the magnitude of v.

the decision is u, v, y. The auxiliary signal v is precomputed and known while u, y become available in real time during the test period. Thus u, y cannot be used in the design of the detection filter. They can only be used in its application.

Failure detection is to be robust with respect to the noise ν. It is assumed that the noise satisfies a quadratic bound. Depending on the quadratic form we will see that this includes both norm bounded noise and model uncertainty. The approach presented here can be applied to more than two models [2] but we focus here on the two model case. It is assumed that there are two possible models, Model 0 and Model 1. For a given auxiliary signal v we define $\mathcal{A}^i(v)$ to be the set of input-output pairs (u, y) consistent with Model i. That is, those pairs which are realizable by that model. Thus

$\mathcal{A}^0(v)$: set of normal input-outputs.

$\mathcal{A}^1(v)$: set of input-outputs when failure occurs.

Failure detection consists of observing the inputs and outputs and deciding to which set they belong. For guaranteed detection we need that

$$\mathcal{A}^0(v) \cap \mathcal{A}^1(v) = \emptyset. \tag{1}$$

In this paper we focus on linear models of different types. Suppose that a given v does not give guaranteed detection. Then we have the situation in Figure 2. As we apply increasing amounts of v, the $\mathcal{A}^i(v)$ move in the direction of the arrows as indicated in Figure 3. If the $\mathcal{A}^i(v)$ are bounded in the appropriate directions, then we reach a multiple of v where the $\mathcal{A}^i(v)$ are tangent as in Figure 4, and after that they are disjoint. Guaranteed detection occurs

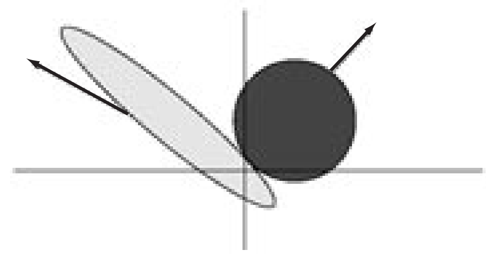

Figure 4: Guaranteed detection with minimal v.

when the sets $\mathcal{A}_i(v)$ are disjoint or tangent. In Figures 2-4 the $\mathcal{A}_i(v)$ stayed a constant size and shape. We will see later that for some choices of noise measure that the $\mathcal{A}_i(v)$ can also grow as v s increases.

The goal then is to have an auxiliary signal v of the smallest possible size and an easily computed function $\mu = h(u, y)$ so that the value of μ will tell us whether failure has occurred. Sections 3 and 4 develop the needed theory for one model. Section 5 then considers auxiliary signal design when there are two models.

3 Static Case

It is instructive to consider first linear systems in the static case. If one then thinks of the matrices as being operators this motivates a number of later developments.

Absence of uncertainty: If we have no uncertainty the model is just

$$y = Gu \tag{2}$$

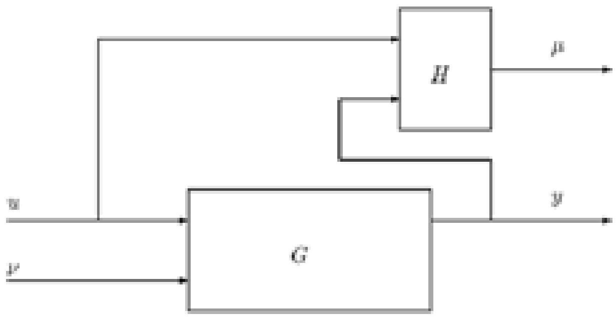

Figure 5: Additive uncertainty.

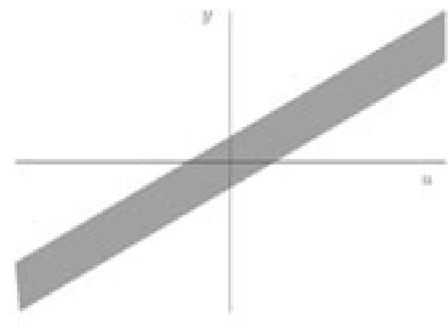

Figure 6: Realizable set for Example 1.

where G is $m \times n$ and 4$\geq m$. For the system (2) we can use a residual-based test:

$$\mu = T(y - Gu) = H \begin{pmatrix} u \\ y \end{pmatrix} \tag{3}$$

where $H = T\begin{pmatrix} -G & I \end{pmatrix}$. Realizability of $\{u, y\}$ is equivalent to μ being zero.

Additive uncertainty: A more realistic case is when we have additive uncertainty

$$y = G_1 u + G_2 \nu \tag{4}$$

where G_1 and G_2 are matrices. We assume G_2 is full row rank. The noise ν is assumed to satisfy (here $\|\cdot\|$ is the usual Euclidian norm):

$$\|\nu\|^2 < d. \tag{5}$$

as illustrated in Figure 5. Then $\{u, y\}$ will be realizable if there exists a ν satisfying (5) and (4).

In order to construct a residual-based test to determine the realizability of a given $\{u, y\}$, we consider the optimization problem:

$$\gamma(u, y) = \min_{\nu} \|\nu\|^2 \tag{6}$$

subject to

$$y = G_1 u + G_2 \nu. \tag{7}$$

The function $\gamma(u, y)$ provides the realizability test.

If $\gamma(u, y) < d$, then $\{u, y\}$ is realizable.
If $\gamma(u, y) \geq d$, then $\{u, y\}$ is not realizable.

We must express γ directly in terms of $\{u, y\}$. The solution to the optimization problem (6), (7) is

$$\gamma(u, y) = \begin{pmatrix} u \\ y \end{pmatrix}^T \begin{pmatrix} -G_1^T \\ I \end{pmatrix} \left(G_2 G_2^T\right)^{-1} \begin{pmatrix} -G_1 & I \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix}.$$

Thus if we let $\mu = H \begin{pmatrix} u \\ y \end{pmatrix}$ where H satisfies

$$H^T H = \begin{pmatrix} -G_1^T \\ I \end{pmatrix} \left(G_2 G_2^T\right)^{-1} \begin{pmatrix} -G_1 & I \end{pmatrix},$$

we have $\gamma(u, y) = \|\mu\|^2$. Then the realizability test becomes $\|\mu\|^2 < d$. If this test does not hold, then we have a failure.

Example 1 *Consider the following simple example*

$$y = u + \nu, \quad \nu^2 < 1.$$

The realizable set $\{u, y\}$ *is given by*

$$\gamma(u, y) = (y - u)^2 < 1$$

and is illustrated in Figure 6.

Slope uncertainty: Of course, in many applications there is uncertainty in the models themselves. To include this type of uncertainty we consider uncertainty in the following form which follows the formulation of [6]:

$$y = (G_{11} + G_{12}\Delta(I - G_{22}\Delta)^{-1}G_{21})u \tag{8}$$

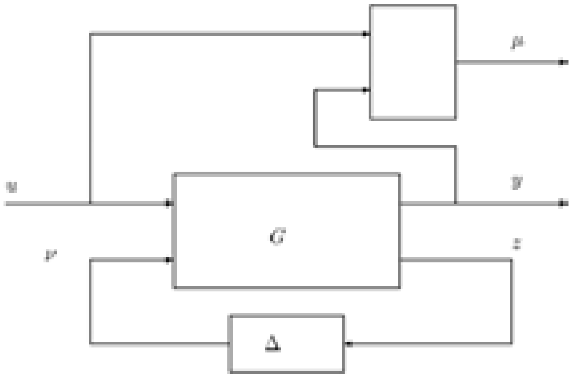

Figure 7: Model uncertainty.

where Δ is any matrix whose maximum singular value $\bar{\sigma}$ satisfies

$$\bar{\sigma}(\Delta) \leq 1. \tag{9}$$

Note that (8) is a perturbation of

$$y = G_{11}u. \tag{10}$$

This corresponds to modeling an uncertain gain as shown in Figure 7. The system of Figure 7 can be re-expressed as

$$y = G_{11}u + G_{12}\nu \tag{11}$$
$$z = G_{21}u + G_{22}\nu \tag{12}$$

with

$$\nu = \Delta z. \tag{13}$$

Note that combining (13) with (12) we get

$$\nu = \Delta(G_{21}u + G_{22}\nu) \tag{14}$$

Solving for ν and substituting into (11) gives (8).

Since (9) holds, we get from (13) that $\|\nu\|^2 \leq \|z\|^2$. Thus the characterization of realizability is

$$\|\nu\|^2 - \|z\|^2 \leq 0. \tag{15}$$

To get this in terms of just $\{u, y\}$ we again set up an optimization problem. Let

$$\gamma(u, y) = \min_{\nu, z} \begin{pmatrix} \nu \\ z \end{pmatrix}^T J \begin{pmatrix} \nu \\ z \end{pmatrix} \tag{16}$$

where

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{17}$$

and $\{v, z\}$ satisfy (11) and (12) for given $\{u, y\}$. These constraints can be expressed as:

$$G_1 \begin{pmatrix} u \\ y \end{pmatrix} = G_2 \begin{pmatrix} \nu \\ z \end{pmatrix} \tag{18}$$

$$G_1 = \begin{pmatrix} -G_{11} & I \\ -G_{21} & 0 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} G_{12} & 0 \\ G_{22} & -I \end{pmatrix}. \tag{19}$$

Let $A_\perp$ be a matrix of maximal full column rank such that $AA_\perp = 0$. That is, $A_\perp$ is a maximal rank right annihilator. Then the solution of the above optimization problem is as follows. Suppose

$$G_{2\perp} J G_{2\perp}^T > 0. \tag{20}$$

Then the solution to optimization problem (16) is

$$\gamma(u, y) = \begin{pmatrix} u \\ y \end{pmatrix}^T G_1^T (G_2 J G_2^T)^{-1} G_1 \begin{pmatrix} u \\ y \end{pmatrix}. \tag{21}$$

Example 2 *Suppose that (8) is*

$$y = (1 + \Delta)u \tag{22}$$

with $|\Delta| \leq 1$*. From (8), it is easy to see that we can take* $G_{22} = 0$ *and* $G_{11} = G_{12} = G_{21} = 1$*. Thus*

$$G_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{23}$$

H *can be obtained using the J-spectral factorization. It is easy to verify that* $(G_2 J G_2^T)^{-1} = J$ *so that we can let* $H = G_1$*, that is,*

$$\mu = \begin{pmatrix} y - u \\ -u \end{pmatrix}. \tag{24}$$

The realizability test (21) then becomes

$$\gamma(u, y) = \mu^T J \mu = y^2 - 2uy \leq 0. \tag{25}$$

To see why this inequality is correct, simply note that $y^2 - 2uy = u^2(\Delta^2 - 1)$*. This inequality defines the set of realizable* $\{u, y\}$ *and is illustrated in Figure 8.*

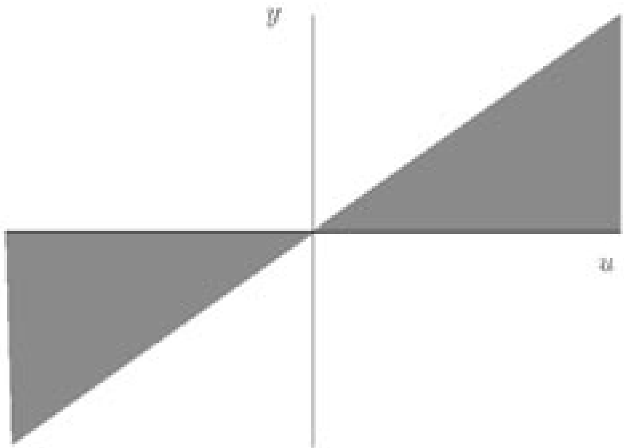

Figure 8: Realizable set for Example 2.

Note that in Figure 6 the set of y's for a given u stays the same size while the set varies in size with u in Figure 8. A similar behavior occurs when designing auxiliary signals later in this paper. The $\mathcal{A}^i(v)$ are fixed size with additive uncertainty and vary in size with model uncertainty.

General uncertainty model: In order to capture both additive and slope uncertainties we consider the more general model

$$G_1 \begin{pmatrix} u \\ y \end{pmatrix} = G_2 \begin{pmatrix} \nu \\ z \end{pmatrix} \tag{26}$$

with G_2 full row rank and with the noise constraint,

$$\begin{pmatrix} \nu \\ z \end{pmatrix}^T J \begin{pmatrix} \nu \\ z \end{pmatrix} < d \tag{27}$$

where $J = \text{diag}(I, -I)$. This is a linear model with a quadratic criteria on the possible noises and perturbations.

Example 3 *Consider the following system*

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{28}$$

with

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} < 1. \tag{29}$$

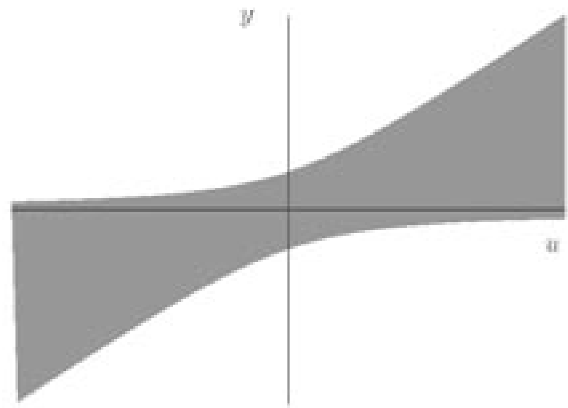

Figure 9: Realizable set for Example 3.

This system is closely related to Example 2. The realizable set is defined by $\gamma(u, y) = y^2 - 2uy < 1$ and is illustrated in Figure 9.

4 Dynamical systems

Continuous and discrete systems are of interest but space allows us only to consider the continuous case. Our approach builds on the formulations in [6]. The static case just discussed leads to the following uncertain dynamical system modeled over $[0, T]$:

$$\dot{x} = Ax + Bu + M\nu, \tag{30}$$

$$z = Gx + Hu, \tag{31}$$

$$y = Cx + Du + N\nu \tag{32}$$

where ν and z are respectively the noise input and noise output representing model uncertainty. N is full row rank.

The constraints on the noises are:

$$x(0)^T P_0^{-1} x(0) + \int_0^s \|\nu\|^2 - \|z\|^2 \, dt < d, \ \forall s \in [0, T]. \tag{33}$$

which is the continuous analogue of (15).

No weights on ν, z are needed since they can be incorporated into the model coefficients. This formulation includes both additive uncertainty and model uncertainty.

For example, as shown in [6] the uncertain system (30), (31) can be used to model

$$\begin{aligned}\dot{x} &= (A + M\Delta G)x + (B + M\Delta H)u,\\ y &= (C + N\Delta G)x + (D + N\Delta H)u\\ &\quad \overline{\sigma}(\Delta(t)) \leq 1\\ &\quad x(0)^T P_0^{-1} x(0) < d\end{aligned}$$

By formulating as an optimization problem, we obtain the realizability test

$$\gamma_s(u, y) < d, \quad \text{for all } s \in [0, T] \tag{34}$$

where

$$\gamma_s(u, y) = \int_0^s \mu^T R^{-1} \mu \, dt \tag{35}$$

and μ is the output of the following system

$$\begin{aligned}\dot{\hat{x}} &= (A - SR^{-1}\bar{C} - PC^T R^{-1}\bar{C})\hat{x}\\ &\quad +(S + P\bar{C}^T)R^{-1}\begin{pmatrix} y - Du \\ -Hu \end{pmatrix} + Bu\\ \mu &= \bar{C}\hat{x} - \begin{pmatrix} y - Du \\ -Hu \end{pmatrix}\end{aligned}$$

with $\hat{x}(0) = x_0$. P is the solution of the Riccati equation

$$\begin{aligned}\dot{P} &= (A - SR^{-1}\bar{C})P + P(A - SR^{-1}\bar{C})^T\\ &\quad -P\bar{C}^T R^{-1}\bar{C}P + Q - SR^{-1}S^T, \ P(0) = P_0\end{aligned}$$

where

$$Q = MM^T, \ S = \begin{pmatrix} MN^T & MJ^T \end{pmatrix}$$
$$R = \begin{pmatrix} NN^T & NJ^T \\ JN^T & -I \end{pmatrix}, \ \bar{C} = \begin{pmatrix} C \\ G \end{pmatrix}.$$

We also assume that

$${N_\perp}^T(I - J^T J)N_\perp > 0, \ \ \forall t \in [0, T].$$

5 Auxiliary signal design

The discussion so far has only addressed how to characterize realizability when there is one model present. We now consider the situation where there are two models and an auxiliary signal v which is to be constructed to aid in identification. We again begin by considering the static case.

5.1 Static case

Let v be the auxiliary input. Suppose that there are two models. By using ν to represent ν and z, and by breaking up the input u into two inputs u and v, we get

$$\begin{pmatrix} X_i & Y_i & Z_i \end{pmatrix}\begin{pmatrix} v \\ u \\ y \end{pmatrix} = H_i \nu_i \tag{36}$$

with the constraints,

$$\nu_i^T J_i \nu_i < 1, \tag{37}$$

for $i = 0, 1$. The case $i = 0$ corresponds to the unfailed system and $i = 1$, the failed system. Here J is a signature matrix (diagonal matrix with $+1$ and -1 entries). Vectors u and y are the on-line measured input and output as before.

We want a test signal v that not only permits perfect identification (1) but also is small in some sense. If the minimization is on the norm of v, our problem becomes that of finding a v of smallest norm for which there exist no solution to (36) and (37), for $i = 0$ and 1 simultaneously.

Example 4 *Consider the following simple systems:*

$$-4v + y = \nu_0 \tag{38}$$
$$\nu_0^2 < 1 \tag{39}$$

and

$$-\begin{pmatrix} 1 \\ 1 \end{pmatrix} v + \begin{pmatrix} 1 \\ 0 \end{pmatrix} y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \nu_1 \tag{40}$$
$$\nu_1^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \nu_1 < 1. \tag{41}$$

Figure 10 shows the input-output sets and the minimal v^* that provides perfect identification.

Non-existence of a solution to (36) and (37) is equivalent to:

$$\sigma(v) \geq 1 \tag{42}$$

where

$$\sigma(v) = \inf_{\nu_0, \nu_1, u, y} \max(\nu_0^T J_0 \nu_0, \nu_1^T J_1 \nu_1) \tag{43}$$

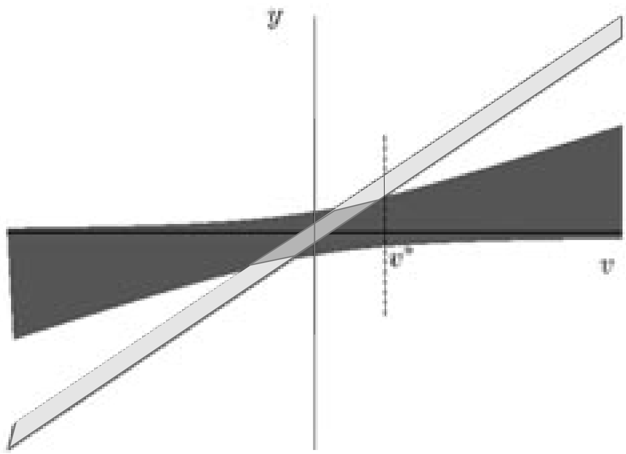

Figure 10: Realizable sets for Example 4 and minimal v.

subject to (36), $i = 0, 1$. It can be shown that $\sigma(v)$ in (43) can be expressed as:

$$\sigma(v) = \max_{\beta \in [0,1]} \inf_{\nu_0, \nu_1, u, y} (\beta \nu_0^T J_0 \nu_0 + (1-\beta) \nu_1^T J_1 \nu_1)$$

subject to

$$\begin{pmatrix} X_0 & Y_0 & Z_0 \\ X_1 & Y_1 & Z_1 \end{pmatrix} \begin{pmatrix} v \\ u \\ y \end{pmatrix} = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_1 \end{pmatrix}. \quad (44)$$

The optimization problem (43) can then now be expressed as follows:

$$\sigma(v) = \max_{\beta} \phi_\beta(v) \quad (45)$$

where

$$\phi_\beta(v) = \min_{\nu_0, \nu_1, u, y} (\beta \nu_0^T J_0 \nu_0 + (1-\beta) \nu_1^T J_1 \nu_1) \quad (46)$$

subject to (44).

Since u and y appear only in the constraint (44), we can eliminate them from the optimization problem by premultiplying (44) with

$$\begin{pmatrix} W_0 & W_1 \end{pmatrix} = \begin{pmatrix} Y_0 & Z_0 \\ Y_1 & Z_1 \end{pmatrix}^{\perp} \quad (47)$$

where $A^\perp$ is a maximal rank left annihilator of A.

So the optimization problem (46) can be expressed as follows:

$$\phi_\beta(v) = \inf_{\nu} \nu^T J_\beta \nu \quad (48)$$

subject to

$$Gv = H\nu \quad (49)$$

where

$$\begin{aligned} G &= \begin{pmatrix} W_0 & W_1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}, \\ H &= \begin{pmatrix} W_0 & W_1 \end{pmatrix} \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix}, \\ \nu &= \begin{pmatrix} \nu_0 \\ \nu_1 \end{pmatrix}, \ J_\beta = \begin{pmatrix} \beta J_0 & 0 \\ 0 & (1-\beta) J_1 \end{pmatrix}. \end{aligned}$$

Proper auxiliary signal and the separability index: An auxiliary signal v is *proper* if the two realizable sets $\mathcal{A}_0(v)$ and $\mathcal{A}_1(v)$ are disjoint. Thus an auxiliary signal v is proper if and only if $\sigma(v) \geq 1$.

Let $\mathcal{V}$ denote the set of proper auxiliary signals v. Suppose that v is measured in an inner product norm so there exists a positive definite matrix Q such that the norm of v is $(v^T Q v)^{1/2}$. Then

$$\gamma^* = \left(\min_{v \in \mathcal{V}} v^T Q v \right)^{-\frac{1}{2}}, \quad (50)$$

is called the *separability index* for positive definite matrix Q. The v realizing the $\min$ is called an optimal proper auxiliary signal.

Using the fact that $\sigma(v)$ is quadratic in v, the problem of finding an optimal auxiliary signal can be formulated as follows:

$$\begin{aligned} \lambda^* &= \max_{v \neq 0} \frac{\sigma(v)}{v^T Q v} = \max_{v \neq 0, \beta \in [0,1]} \frac{\phi_\beta(v)}{v^T Q v} \\ &= \max_{v \neq 0, \beta \in [0,1]} \frac{v^T V_\beta v}{v^T Q v} \end{aligned} \quad (51)$$

for some symmetric matrix V_β. We are assuming here that $H_\perp^T J_\beta H_\perp > 0$. Let

$$\lambda_\beta = \max_{v \neq 0} \frac{v^T V_\beta v}{v^T Q v}. \quad (52)$$

Then clearly $\lambda^* = \max_\beta \lambda_\beta$. But λ_β is the largest λ such that

$$v^T V_\beta v - \lambda v^T Q v = 0. \tag{53}$$

So if $\lambda^* > 0$, then the set of proper auxiliary signals is not empty and any optimal proper auxiliary signal is a solution of (53) satisfying

$$v^T Q v = \frac{1}{\lambda^*}. \tag{54}$$

If $\lambda^* \leq 0$, then the set of proper auxiliary signals is empty. If $\lambda^* > 0$, then the separability index is given by

$$\gamma^* = \frac{1}{\sqrt{\lambda^*}}. \tag{55}$$

Example 5 *For Example 4, $\phi_\beta(v)$ is:*

$$\inf_{\nu_0,\nu_{10},\nu_{11}} \begin{pmatrix} \nu_0 \\ \nu_{10} \\ \nu_{11} \end{pmatrix}^T \begin{pmatrix} \beta & & \\ & 1-\beta & \\ & & -(1-\beta) \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_{10} \\ \nu_{11} \end{pmatrix}$$

subject to

$$\begin{pmatrix} -4 & 1 \\ -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \nu_0 \\ \nu_{10} \\ \nu_{11} \end{pmatrix}.$$

We can show that $\phi_\beta(v)$ is defined for all $\beta \in [0,1]$ and is given by $\phi_\beta(v) = (9\beta - 1)(1-\beta)v^2$. So $\sigma(v) = \max_{\beta\in[0,1]} \phi_\beta(v)$ can easily be computed in this case by setting the derivative of $\phi_\beta(v)$ with respect to β to zero to get $\beta^* = 5/9$. This gives $\sigma(v) = \frac{16}{9}v^2$ and thus $v^* = 3/4$.

Alternative solution: Instead of performing two optimizations, we can combine them to have a one step solution. Suppose that $H_\perp^T J_\beta H_\perp > 0$. Then λ_β corresponds to the largest λ such that

$$\det(\lambda H J_\beta^{-1} H^T - GG^T) = 0. \tag{56}$$

Computing

$$\lambda^* = \max_{\beta\in[a,b]} \lambda_\beta \tag{57}$$

a proper auxiliary signal exists if and only if $\lambda^* > 0$. Then an optimal auxiliary signal is

$$v^* = K\zeta^* \tag{58}$$

where $K = G^l H J_\beta^{-1} H^T$, G^l is any left-inverse of G, $\zeta^* = \alpha\zeta$ where ζ is any non-zero vector satisfying

$$(\lambda^* H J_{\beta^*}^{-1} H^T - GG^T)\zeta = 0 \tag{59}$$

where β^* is a value of β maximizing in (57), and

$$\alpha = \frac{1}{\sqrt{\lambda^* \zeta^T K^T Q K \zeta}}. \tag{60}$$

5.2 Continuous-time case

By using ν to represent ν and z, and by breaking up the input u into two inputs u and v, we obtain the following models:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i v + \bar{B}_i u + M_i \nu_i, && (61) \\ E_i y &= C_i x_i + D_i v + \bar{D}_i u + N_i \nu_i && (62) \end{aligned}$$

where $i = 0, 1$ correspond to normal and failed systems. N_i's are assumed to have full row rank. The noise constraints are

$$\begin{aligned} \mathcal{S}_i(v,s) &= x_i(0)^T P_{i0}^{-1} x_i(0) \\ &\quad + \int_0^s \nu_i^T J_i \nu_i \, dt < 1, \ \forall s \in [0,T], \end{aligned} \tag{63}$$

where the J_i's are signature matrices.

Non-existence of a solution to (61), (62) and (63) is equivalent to:

$$\sigma(v,s) \geq 1 \tag{64}$$

where

$$\sigma(v,s) = \inf_{\substack{\nu_0,\nu_1,u,y \\ x_0,x_1}} \max(\mathcal{S}_0(v,s), \mathcal{S}_1(v,s)), \tag{65}$$

subject to (61)-(62), $i = 0, 1$. As in the static case, we can reformulate (65) as:

$$\sigma(v,s) = \max_{\beta\in[0,1]} \phi_\beta(v,s) \tag{66}$$

where

$$\phi_\beta(v,s) = \inf_{\substack{\nu_0,\nu_1,u,y \\ x_0,x_1}} \beta\mathcal{S}_0(v,s) + (1-\beta)\mathcal{S}_1(v,s)$$

subject to (61)-(62), $i = 0, 1$.

We consider here only the case where there is no u. That is, all the inputs are used as an auxiliary signal for failure detection. Let

$$x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix},\ \nu = \begin{pmatrix} \nu_0 \\ \nu_1 \end{pmatrix},\ A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix},$$

$$M = \begin{pmatrix} M_0 & 0 \\ 0 & M_1 \end{pmatrix},\ N = \begin{pmatrix} F_0 N_0 & F_1 N_1 \end{pmatrix},$$

$$D = F_0 D_0 + F_1 D_1, C = \begin{pmatrix} F_0 C_0 & F_1 C_1 \end{pmatrix},$$

$$B = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}, P_\beta^{-1} = \begin{pmatrix} \beta P_{0,0}^{-1} & 0 \\ 0 & (1-\beta) P_{1,0}^{-1} \end{pmatrix},$$

$$J_\beta = \begin{pmatrix} \beta J_0 & 0 \\ 0 & (1-\beta) J_1 \end{pmatrix},$$

$$\begin{pmatrix} F_0 & F_1 \end{pmatrix} = \begin{pmatrix} E_0 \\ E_1 \end{pmatrix}^\perp.$$

Then we can reformulate the problem (65) as:

$$\phi_\beta(v,s) = \inf_{\nu,x} x(0)^T P_\beta^{-1} x(0) + \int_0^T \nu^T J_\beta \nu \, dt$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bv + M\nu \\ 0 &= Cx + Dv + N\nu. \end{aligned}$$

Construction of an optimal proper auxiliary signal: For the sake of simplicity of presentation, we start by considering that the optimality criterion used for the auxiliary signal is just that of minimizing its L_2 norm. Thus the problem to solve is:

$$\min_v |||v|||^2, \quad \text{subject to} \max_{\substack{\beta \in [0,1] \\ s \in [0,T]}} \phi_\beta(v,s) \geq 1 \qquad (67)$$

where

$$|||v|||^2 = \int_0^T \|v\|^2 \, dt.$$

The reason we take the $\max$ over s is that the maximum value of $\phi_\beta(v,s)$ does not always occur at $s = T$. It does in most cases though.

As we did in the static case, we reformulate the optimization problem as follows

$$\lambda_{\beta,s} = \max_{v \neq 0} \frac{\phi_\beta(v,s)}{|||v|||^2}. \qquad (68)$$

Suppose for some $\beta \in [0,1]$, that we have ${N_\perp}^T J_\beta N_\perp > 0$, $\forall\, t \in [0,T]$, and that the Riccati equation

$$\begin{aligned} \dot{P} &= (A - SR^{-1}C)P + P(A - SR^{-1}C)^T \\ &\quad - PC^T R^{-1} CP + Q - SR^{-1} S^T, \quad P(0) = P_\beta \end{aligned}$$

where

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} J_\beta^{-1} \begin{pmatrix} M \\ N \end{pmatrix}^T \qquad (69)$$

has a solution on [0,T]. The values of Q, S, R depend on β. Then $\lambda_{\beta,s}$ is the smallest λ for which the Riccati equation

$$\begin{aligned} \dot{P} &= (A - S_\lambda R_\lambda^{-1} C)P + P(A - S_\lambda R_\lambda^{-1} C)^T \\ &\quad - PC^T R_\lambda^{-1} CP + Q_\lambda - S_\lambda R_\lambda^{-1} S_\lambda^T, \ P(0) = P_\beta \end{aligned}$$

where

$$\begin{pmatrix} Q_\lambda & S_\lambda \\ S_\lambda^T & R_\lambda \end{pmatrix} = \begin{pmatrix} M & B \\ N & D \end{pmatrix} \begin{pmatrix} J_\beta & 0 \\ 0 & -\lambda I \end{pmatrix}^{-1} \begin{pmatrix} M & B \\ N & D \end{pmatrix}^T$$

has a solution on $[0, s]$. The values of $Q_\lambda, S_\lambda, R_\lambda$ depend on β as well as λ.

This result allows us to compute

$$\lambda_\beta = \max_s \lambda_{\beta,s}$$

which can be used to compute

$$\lambda^* = \max_\beta \lambda_\beta.$$

This gives the separability index as

$$\gamma^* = \frac{1}{\sqrt{\lambda^*}}.$$

and is used to compute the optimal auxiliary signal v^*.

Construction of optimal auxiliary signal: Assume β^* (an optimal β) and λ^* are computed. Let (x, ζ)

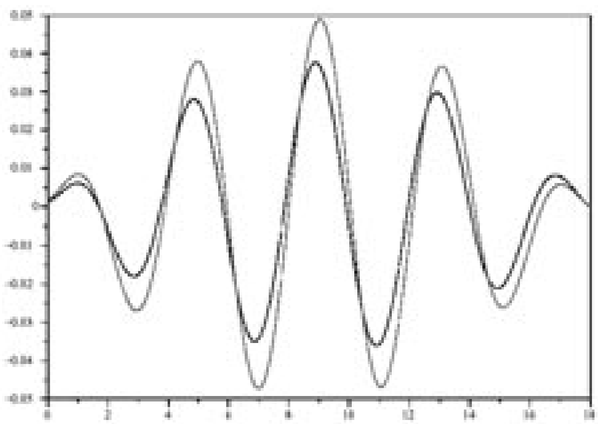

Figure 11: Typical minimum proper v.

be any non-zero solution of the two-point boundary-value system:

$$\frac{d}{dt}\begin{pmatrix} x \\ \zeta \end{pmatrix} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}\begin{pmatrix} x \\ \zeta \end{pmatrix}$$

with boundary conditions:

$$\begin{aligned} x(0) &= P_{\beta^*}\zeta(0) \\ \zeta(T) &= 0. \end{aligned}$$

where

$$\begin{aligned} \Omega_{11} &= -\Omega_{22}^T = A - S_{\lambda^*}R_{\lambda^*}^{-1}C \\ \Omega_{12} &= Q_{\lambda^*} - S_{\lambda^*}R_{\lambda^*}^{-1}S_{\lambda^*}^T \\ \Omega_{21} &= C^T R_{\lambda^*}^{-1}C \end{aligned}$$

This boundary value problem is not well-posed and has non trivial solutions (x^*, ζ^*). An optimal auxiliary signal is (for simplicity suppose $D = 0$)

$$v^* = \alpha B^T \zeta^*$$

where α is a constant such that $|||v^*||| = 1/\gamma^*$. A typical auxiliary signal is given in Figure 11. The shape of the v can vary depending on the models, the size of the weight on the initial condition perturbation, and the length of the interval. Several more examples are in [2] and the other references.

6 Conclusion

We have described a framework for multi-model identification and shown the form it takes for two specific types of problems. In this approach the auxiliary signal design is done off line. It easy to implement the algorithms in Scilab [1] or other CAD packages. The on-line detection test can be implemented efficiently using "Kalman type" filters or hyperplane tests but space does not permit discussing this here.

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