Pencil Algorithms and Observer Design

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Abstract

We give canonical forms for a general singular matrix pencil from which several observer design problems are easily solved. The geometrical basis of these forms is emphasized. Algorithms for the robust computation of the needed subspaces are given.

1. Introduction

Many of the various canonical forms, such as the Kronecker form, are difficult to compute and are often presented in a nongeometric fashion. In this paper we define certain subspaces and show how they can be computed in a numerically robust fashion. Then we show how these subspaces and algorithms are related to the observer design algorithm discussed in [4].

2. Geometrical Theory and Algorithms

The range and nullspace (kernel) of a linear transformation E will be denoted $\mathcal{K}(E)$ and $\mathcal{R}(E)$ respectively. In the sequel we will often use the same notation to denote a linear map and its matrix representation. When no confusion is possible we also use the matrix notation to denote a linear map and its range. In particular, $E^{-1}(F)$ will denote the subspace $E^{-1}(\mathcal{R}(F))$. A matrix will be called a basis if its columns are a basis for the range.

Consider a pair of linear maps (E, F): $\mathcal{X} \to \mathcal{Y}$ or, equivalently, a linear pencil $sE - F$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional real vector spaces of dimension p and q respectively. The pair of subspaces $(\mathcal{Z} \subset \mathcal{X}, \mathcal{Q} \subset \mathcal{Y})$ is (E, F)-invariant if $E(\mathcal{Z}) \subset \mathcal{Q}$ and $F(\mathcal{Z}) \subset \mathcal{Q}$. Simple examples of such pairs are $\mathcal{Z} = \mathcal{K}(E), \mathcal{Q} = F(\mathcal{K}(E))$ and $\mathcal{Z} = F^{-1}(E)$, $\mathcal{Q} = \mathcal{R}(E)$. In matrix terms, this means that if $X = \left[\begin{array}{ccc} x_1, \ldots, x_p \end{array}\right]$ and $Y = \left[\begin{array}{ccc} y_1, \ldots, y_q \end{array}\right]$ are matrices whose columns are basis of $\mathcal{X}$ and $\mathcal{Y}$ such that columns 1 to k_1 of X span $\mathcal{Z}$ and columns 1 to k_2 of Y span $\mathcal{Q}$, then the pair of linear maps (E, F) is represented in these bases by the matrices:

$$(Y^{-1}EX, Y^{-1}FX) = \left(\left[\begin{array}{cc} \overline{E} & * \\ 0 & \tilde{E} \end{array}\right], \left[\begin{array}{cc} \overline{F} & * \\ 0 & \tilde{F} \end{array}\right] \right) \quad (1)$$

Here, 0 is $(q - k_2) \times k_1$. Note that the restriction maps $\overline{E} : \mathcal{Z} \ni x \to Ex \in \mathcal{Q}$ and $\overline{F} : \mathcal{Z} \ni x \to Fx \in \mathcal{Q}$ are well defined. $\tilde{E}$ (resp. $\tilde{F}$) is the matrix representation of the linear map: $\mathcal{X}/\mathcal{Z} \ni x(mod\mathcal{Z}) \to Ex(mod\mathcal{Q}) \in \mathcal{Y}/\mathcal{Q}$ (resp. $x(mod\mathcal{Z}) \to Fx(mod\mathcal{Q})$) in suitable basis. If π_1 (resp. π_2) denotes the canonical projection $x \to x(mod\mathcal{Z})$ (resp. $y \to y(mod\mathcal{Q})$), $\tilde{E}$ (resp. $\tilde{F}$) is defined by $\tilde{E}\pi_1 = \pi_2 E$ (resp. $\tilde{F}\pi_1 = \pi_2 F$). We call the pair $\tilde{E}, \tilde{F}$ the quotient pair associated with the pair $(\mathcal{Z}, \mathcal{Q})$.

2.1. Useful lemmas

In this Section we give some elementary lemmas and associated constructive algorithms that are used as building blocks for our observer construction. From a strictly mathematical point of view row (or column) compression can be done just by left (or right) multiplication by an orthogonal matrix. However, in order to compute this orthogonal transformation in a robust manner, say by a QR algorithm, it is necessary to allow for column (or row pivoting.) Thus the permutation matrices could be deleted from most of the theorems that follow but we include them since the algorithms would use permutation matrices.

First, we have the following simple lemma:

Lemma 2.1 *If the pair* $(\mathcal{Z}, \mathcal{Q})$ *is* (E, F)*-invariant and the pair* $(\tilde{\mathcal{Z}}, \tilde{\mathcal{Q}})$ *is* $(\tilde{E}, \tilde{F})$*-invariant, then the pair* $(\pi_1^{-1}\tilde{\mathcal{Z}}, \pi_2^{-1}\tilde{\mathcal{Q}})$ *is* (E, F)*-invariant.*

Let Z_1 and Q_1 be orthogonal basis of $\mathcal{X}$ and $\mathcal{Y}$ such that the first k_1 columns of Z span $\mathcal{Z}$ and the first k_2

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columns of Q_1 span $\mathcal{Q}$. Then $Q_1^T E Z_1 = \begin{bmatrix} \overline{E} & * \\ 0 & \tilde{E} \end{bmatrix}$ and $Q_1^T F Z_1 = \begin{bmatrix} \overline{F} & * \\ 0 & \tilde{F} \end{bmatrix}$. Now, if Q_2 and Z_2 are orthogonal matrices such that $Q_2^T \tilde{E} Z_2 = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, $Q_2^T \tilde{F} Z_2 = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ (where the zero block has k_1' columns), then $Q = Q_1 diag(I, Q_2)$ and $Z = Z_1 diag(I, Z_2)$ are orthogonal. Columns 1 to $k_1 + k_1'$ of Z span ${\pi_1}^{-1}\tilde{\mathcal{Z}}$ and columns 1 to $k_2 + k_2'$ of Q^T span ${\pi_2}^{-1}\tilde{\mathcal{Q}}$. This construction gives a numerically stable matrix implementation to the preceding lemma.

The following lemma shows how to numerically calculate the subspace $M^{-1}(N)$.

Lemma 2.2 *Given two linear maps $M : \mathcal{X}_1 \to \mathcal{Y}$ and $N : \mathcal{X}_2 \to \mathcal{Y}$ there exist two orthogonal matrices Q, Z_1 and a permutation matrix P such that:*

$$Q \begin{bmatrix} MZ_1 & NP \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} * & * \\ 0 & \overline{M} \end{bmatrix} & \begin{bmatrix} \overline{N} \\ 0 \end{bmatrix} \end{bmatrix}$$

where $\overline{N}$ has full row rank, $rank(\overline{N}) = rank(N)$, $\overline{M}$ has full column rank. If $k_1 = dim(\mathcal{X}_1) - rank(\overline{M})$, then $dim(M^{-1}(N)) = k_1$ and the first k_1 columns of Z_1 span $M^{-1}(N)$.

<u>Proof</u> Define Q to be an orthogonal row compression of N obtained by the QR decomposition of N with column pivoting. That is, $QNP = \begin{bmatrix} \overline{N} \\ 0 \end{bmatrix}$ where $\overline{N}$ has full row rank. Then, with $k_0 = rank(N) = rank(\overline{N})$, the first k_0 columns of Q^T span $\mathcal{R}(N)$ and the vector x belongs to $\mathcal{R}(N)$ iff $Qx = \begin{bmatrix} * \\ 0 \end{bmatrix}$ (with partition compatible with $\begin{bmatrix} \overline{N} \\ 0 \end{bmatrix}$). Now, if $\mathcal{Y}$ is equipped with the orthogonal basis Q^T, the matrix representation of $\begin{bmatrix} M & N \end{bmatrix}$ is

$$\begin{bmatrix} QM & QNP \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} & \begin{bmatrix} \overline{N} \\ 0 \end{bmatrix} \end{bmatrix}$$

The vectors of $\mathcal{R}(M)$ which belong to $\mathcal{R}(N)$ are clearly obtained by a column compression of M_2. If Z_1 is an orthogonal basis which realizes this column compression, i.e. $M_2 Z_1 = \begin{bmatrix} 0 & \overline{M} \end{bmatrix}$ with $rank(\overline{M}) = rank(M_2) = dim(\mathcal{X}_1) - k_1$, then the first k_1 columns of Z_1 have M-image $\begin{bmatrix} * \\ 0 \end{bmatrix}$ and they span $M^{-1}(N)$. □

In general P and Q are not useful and we can implement the following algorithm

Algorithm `[Z,k]=iminv(M,N)`

of rows.

Outputs: Z, orthogonal matrix (order equal to the number of columns of M) and k, integer. The first k columns of Z span $M^{-1}(N)$. □

Given two subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{Y}$, we have the usual lattice:

Lemma 2.3 *Given two linear maps $M : \mathcal{X}_1 \to \mathcal{Y}$ and $N : \mathcal{X}_2 \to \mathcal{Y}$ there exists two orthogonal matrices Q and Z_2 and a permutation matrix P such that:*

$$Q \begin{bmatrix} MP & NZ_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \overline{M}_1 \\ \overline{M}_2 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \overline{N}_1 & * \\ 0 & * \\ 0 & \overline{N}_2 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

where $\begin{bmatrix} \overline{M}_1 \\ \overline{M}_2 \end{bmatrix}$ has full row rank, $\overline{N}_1$ has full row rank and $\overline{N}_2$ is square and nonsingular. If k_I (resp. k_M, resp. k_{MN}) denotes the row dimension of the first block (resp. the first two blocks, resp. the first three blocks) in this partition, then the first k_I columns of Q^T span $\mathcal{R}(M) \cap \mathcal{R}(N)$ (resp. $\mathcal{R}(M)$, resp. $\mathcal{R}\left(\begin{bmatrix} M & N \end{bmatrix}\right) = \mathcal{R}(M) + \mathcal{R}(N)$)

<u>Proof</u> Performing the QR factorization of M we obtain, with Q_1 an orthogonal matrix and P_1 a permutation matrix , the row compression of $\begin{bmatrix} M & N \end{bmatrix}$ as

$$Q_1[MP_1 \quad N] = \begin{bmatrix} \overline{M} & \widetilde{N}_1 \\ 0 & \widetilde{N}_2 \end{bmatrix}$$

where $\overline{M}$ has full row rank. Then, by a QR row compression of $\widetilde{N}_2$ we have

$$Q_2 Q_1 [MP_1 \quad NP_2] = \begin{bmatrix} \overline{M} & \overline{N} \\ 0 & N_2 \\ 0 & 0 \end{bmatrix}$$

Here $\overline{M}$ has full row rank k_M, $\begin{bmatrix} \overline{M} & \overline{N} \\ 0 & N_2 \end{bmatrix}$ has full row rank k_{MN} (in particular N_2 has full row rank). At this stage, the first k_M columns of $(Q_2 Q_1)^T$ span $\mathcal{R}(M)$ and the first k_{MN} columns of $(Q_2 Q_1)^T$ span $\mathcal{R}([M \quad N])$. A column compression of N_2 yields $\begin{bmatrix} N_1 \\ N_2 \\ 0 \end{bmatrix} Z_2 = \begin{bmatrix} N_{11} & * \\ 0 & N_{21} \\ 0 & 0 \end{bmatrix}$ where N_{21} is square and nonsingular. By Lemma 2.2, the first columns of Z_2 span $N^{-1}(M)$ and thus $\mathcal{R}\left(\begin{bmatrix} N_{11} \\ 0 \\ 0 \end{bmatrix}\right) = N(N^{-1}(M))$

sion of $\begin{bmatrix} N_{11} \\ 0 \\ 0 \end{bmatrix}$ (obtained by compressing N_{11} i.e. $Q_3 \begin{bmatrix} N_{11} \\ 0 \\ 0 \end{bmatrix} = \left[\begin{array}{c} \overline{N}_1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right]$), where $\overline{N}_1$ has full row rank, the first columns of Q_3^T span $\mathcal{R}(N) \cap \mathcal{R}(M)$ and Q is obtained by $Q = Q_3 Q_2 Q_1$. □

In general P and Z_2 are not useful and we can implement the following algorithm:

Algorithm `[Q,`k_I`,`k_M`,`k_{MN}`]=spans(M,N)`

Inputs: M and N two matrices with the same number of rows.

Outputs: Q, an orthogonal matrix and three integers $k_I \leq k_M \leq k_{MN}$ such that the first k_I columns of Q^T span $\mathcal{R}(M) \cap \mathcal{R}(N)$, the first k_M columns of Q^T span $\mathcal{R}(M)$ and the first k_{MN} columns of Q^T span $\mathcal{R}(M) + \mathcal{R}(N)$. □

One use of this algorithm is that it provides a simple test to check if a given vector, say x, belongs to $\mathcal{R}(M) \cap \mathcal{R}(N)$ (resp. $\mathcal{R}(M)$, resp. $\mathcal{R}(M) + \mathcal{R}(N)$): it is sufficient to form the product Qx and examine its zero entries.

2.2. Basic subspaces

Many subspaces associated with matrix pencils have been introduced in the literature [3]. We consider here two basic pairs of invariant subspaces. The pair ($\mathcal{Z}_{\epsilon f}$, $\mathcal{Q}_{\epsilon f}$) is defined as the limit as $k \to \infty$ of the decreasing sequence:

$$\mathcal{Z}_{\epsilon f}(k+1) = F^{-1}(E\mathcal{Z}_{\epsilon f}(k)) \qquad \mathcal{Z}_{\epsilon f}(0) = \mathcal{X}$$

$$\mathcal{Q}_{\epsilon f}(k+1) = E(F^{-1}\mathcal{Q}_{\epsilon f}(k)) \qquad \mathcal{Q}_{\epsilon f}(0) = \mathcal{Y}$$

We note that $\mathcal{Q}_{\epsilon f} = E(\mathcal{Z}_{\epsilon f})$ and $\mathcal{Z}_{\epsilon f} = F^{-1}(\mathcal{Q}_{\epsilon f})$ (which implies $F(\mathcal{Z}_{\epsilon f}) = \mathcal{R}(F) \cap \mathcal{Q}_{\epsilon f}$).

The second pair $(\mathcal{Z}_{\epsilon\infty}, \mathcal{Q}_{\epsilon\infty})$ is defined as the limit as $k \to \infty$ of the increasing sequences

$$\mathcal{Z}_{\epsilon\infty}(k+1) = E^{-1}(F\mathcal{Z}_{\epsilon\infty}(k)) \qquad \mathcal{Z}_{\epsilon\infty}(0) = 0$$

$$\mathcal{Q}_{\epsilon\infty}(k+1) = F(E^{-1}\mathcal{Q}_{\epsilon\infty}(k)) \qquad \mathcal{Q}_{\epsilon\infty}(0) = 0$$

We have $\mathcal{Q}_{\epsilon\infty}(k+1) = F(\mathcal{Z}_{\epsilon\infty}(k))$ which implies $\mathcal{Q}_{\epsilon\infty} = F(\mathcal{Z}_{\epsilon\infty})$ and $\mathcal{Z}_{\epsilon\infty} = E^{-1}(\mathcal{Q}_{\epsilon\infty})$. In particular, $E(\mathcal{Z}_{\epsilon\infty}) = \mathcal{R}(E) \cap \mathcal{Q}_{\epsilon\infty}$.

Theorem 2.1 *Given a pair of linear maps (E, F), the pair of invariant subspaces $(\mathcal{Z}_{\epsilon f}, \mathcal{Q}_{\epsilon f})$ is such that its associated restriction pencil $(\overline{E}_{\epsilon f}, \overline{F}_{\epsilon f})$*

- *has the same right minimal indices and finite zeros as (E, F)*

Proof We can assume without restriction that the pair (E, F) is in Kronecker block diagonal canonical form. We examine, in that order, the behavior of the sequences $\mathcal{Q}_{\epsilon f}(1) = \mathcal{R}(E)$, $\mathcal{Z}_{\epsilon f}(1) = F^{-1}\mathcal{Q}_{\epsilon f}(1)$, $\mathcal{Q}_{\epsilon f}(2) = E(\mathcal{Z}_{\epsilon f}(1))$, $\mathcal{Z}_{\epsilon f}(2) = F^{-1}\mathcal{Q}_{\epsilon f}(2)$, ...

Let $n_0^{(\infty)}, n_1^{(\infty)}, \ldots, n_p^{(\infty)}$ be the number of Jordan blocks of respective dimensions $1, 2, \ldots, p+1$ at ∞. Let $r_0, r_1, \ldots, r_d$ be the respective number of "r.m.i blocks" of dimensions $(1\times 0), (2\times 1), \ldots, (d+1\times d)$ corresponding respectively to the r.m.i's $\eta_0 = 0, \ldots, \eta_l = l$.

At the first step, $\mathcal{Q}_{\epsilon f}(1) = \mathcal{R}(E)$ is obtained by row compressing E: the only zero rows are found in each Jordan block at infinity and each "r.m.i" block at position say $l_1, l_2, ...l_p$. Then $\mathcal{Z}_{\epsilon f}(1) = F^{-1}(\mathcal{Q}_{\epsilon f}(1))$ is obtained (see lemma 2.2) by selecting columns of F which have zero entries at row l_i. At this step we see that we discard $\sum_{j=k}^{p} n_j^{(\infty)} + \sum_{i=k}^{d} r_i$ rows and columns. Then we continue with a reduced pencil where the dimension of each infinite and "r.m.i" block has been reduced by one. We determine $\mathcal{Q}_{\epsilon f}(2) = E(\mathcal{Z}_{\epsilon f}(1))$ by a row compression of the E matrix restricted to the appropriate selected columns and find $\mathcal{Z}_{\epsilon f}(2)$ as in the first step. It is easily seen that we have:

$$dim(\mathcal{Q}_{\epsilon f}(k)) - dim(\mathcal{Q}_{\epsilon f}(k+1)) = \sum_{j=k}^{p}(n_j^{(\infty)} - k) + \sum_{i=k}^{d}(r_i - k)$$

and the algorithm stops when the reduced pencil has no infinite nor "r.m.i" block.

Theorem 2.2 *Given a pair of linear maps (E, F), the pair of invariant subspaces $(\mathcal{Z}_{\epsilon\infty}, \mathcal{Q}_{\epsilon\infty})$ is such that its associated restriction pencil $(\overline{E}_{\epsilon\infty}, \overline{F}_{\epsilon\infty})$*

- *has no right minimal indices and no finite zeros*
- *has the same right minimal indices and infinite zeros as (E, F)*

Proof Let $n_0^{(\infty)}, n_1^{(\infty)}, \ldots, n_p^{(\infty)}$ be the number of Jordan blocks of respective dimensions $1, 2, \ldots, p+1$ at ∞. Let $c_0, c_1, \ldots, c_g$ be the respective number of "c.m.i blocks" of dimensions $(0\times 1), (1\times 2), (g\times g+1)$ associated with $\epsilon_0 = 1, \ldots, \epsilon_g = g$. We now examine, in that order, the nested sequence $\mathcal{Z}_{\epsilon\infty}(1) = \mathcal{K}(E)$, $\mathcal{Q}_{\epsilon\infty}(1) = F(\mathcal{Z}_{\epsilon\infty}(1))$, $\mathcal{Z}_{\epsilon\infty}(2) = E^{-1}(\mathcal{Q}_{\epsilon\infty}(1))$, ... We obtain here:

$$dim(\mathcal{Z}_{\epsilon\infty}(k+1)) - dim(\mathcal{Z}_{\epsilon\infty}(k)) = \sum_{j=k}^{g} c_j + \sum_{i=k}^{p} n_i^{(\infty)}$$

and column vector in each Jordan block at infinity and in each "c.m.i" block and the algorithms stops when all such vectors have been selected. □

Note that, using lemma 2.2, the recursions which give the pairs $(\mathcal{Z}_{\epsilon f}, \mathcal{Q}_{\epsilon f})$ and $(\mathcal{Z}_{\epsilon\infty}, \mathcal{Q}_{\epsilon\infty})$ are easily implemented as matrix algorithms in which the $Q(k)$ and $Z(k)$ are calculated together. The interpretation of $\mathcal{Z}_{\epsilon f}(k)$ is as follows:

$$\mathcal{Z}_{\epsilon f}(k) = \mathcal{F}_k^{-1}(\mathcal{E}_k)$$

$$\{x_0 | \exists x_1, x_2, \ldots, x_k \qquad Ex_{n+1} = Fx_n \text{ for } n = 0, \ldots, k\}$$

where

$$\mathcal{F}_k = \begin{bmatrix} -F \\ 0 \\ 0 \\ \vdots \end{bmatrix} \qquad \mathcal{E}_k = \begin{bmatrix} E & 0 & 0 & \ldots \\ -F & E & 0 & \ldots \\ 0 & -F & E & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Indeed, it is easy to see that the ϵf recursion can be interpreted as a recursive implementation of Lemma (2.2), taking into account the particular structure of $\mathcal{F}_k$ and $\mathcal{E}_k$. Note that, in particular, the system with inputs $Ex_{n+1} = Fx_n + Gu_n$ is equivalent to $PEx_{n+1} = PFx_n + PGu_n$ i.e. $\begin{bmatrix} \overline{E} \\ \tilde{E} \end{bmatrix} x_{n+1} = \begin{bmatrix} \overline{F} \\ \tilde{F} \end{bmatrix} x_n + \begin{bmatrix} \overline{G} \\ 0 \end{bmatrix} u_n$ where P is a row compression of G. Since $\overline{G}$ has full row rank, the subspace $\mathcal{V} = \mathcal{Z}_{\epsilon f}(\tilde{E}, \tilde{F}) = \mathcal{Z}_{\epsilon f}(\pi E, \pi F)$ (where π is the canonical projection $u \to u(modG)$) is the set of admissible x_0 for this system. Since $\pi F^{-1}(\pi E)X = F^{-1}(EX + \mathcal{R}(G))$, $\mathcal{V}$ is also given as the limit of $\mathcal{V}_0 = \mathcal{X}, \quad \mathcal{V}_{k+1} = F^{-1}(E\mathcal{V}_k + \mathcal{R}(G))$ [1]. Similar interpretations can be given for the $\epsilon\infty$ algorithm.

More generally, it is convenient to implement the ϵf and $\epsilon\infty$ algorithms on an appropriate subset of rows or columns, i.e. "localizing" the calculated subspaces in given subspaces as done above.

2.3. Lattice

Applying Lemma 2.3 twice to the pairs $(\mathcal{Z}_{\epsilon f}, \mathcal{Q}_{\epsilon f})$ and $(\mathcal{Z}_{\epsilon\infty}, \mathcal{Q}_{\epsilon\infty})$ we obtain the subspaces $\mathcal{Z}_\epsilon = \mathcal{Z}_{\epsilon f} \cap \mathcal{Z}_{\epsilon\infty}$, $\mathcal{Z}_{\epsilon f}$, $\mathcal{Z}_{\epsilon\infty}$, $\mathcal{Z}_{\epsilon f\infty} = \mathcal{Z}_{\epsilon f} + \mathcal{Z}_{\epsilon\infty}$ in $\mathcal{X}$ and $\mathcal{Q}_\epsilon = \mathcal{Q}_{\epsilon f} \cap \mathcal{Q}_{\epsilon\infty}$, $\mathcal{Q}_{\epsilon f}$, $\mathcal{Q}_{\epsilon\infty}$, $\mathcal{Q}_{\epsilon f\infty} = \mathcal{Q}_{\epsilon f} + \mathcal{Q}_{\epsilon\infty}$ in $\mathcal{Y}$. This yields the block triangular decomposition

$$\begin{bmatrix} F_\epsilon(s) & * & * & * \\ 0 & F_\infty(s) & * & * \\ 0 & 0 & F_f(s) & * \\ 0 & 0 & 0 & F_\eta(s) \end{bmatrix} \qquad (2)$$

This decompostion is well known. It is given, for example, in [5] where a different approach is used (which allows to compute the fine structure of the Kronecker changed, but note that, in this upper block triangular form, the ϵ block must appear in the first position and the η block must appear in the fourth position. It is clear that we can use the approach in lemma 2.1 for finding specific invariant pairs for the pencil (E, F). For instance, using the ordered Schur form for the pencil $F_f(s)$ (which has only finite zeros, i.e. equivalent to $sI - A_f$) we can find the pair which contains the ϵ part and the stable zeros of the pencil. For a regular pencil, two subspaces associated with disjoint finite zeros do not intersect: in the general case considered here their intersection is $\mathcal{Z}_\epsilon$.

2.4. Infinity block

We can define two subspaces associated with the "static" and "dynamic" zeros at infinity [7].

Theorem 2.3 *$\mathcal{Z}_{\epsilon s} = \mathcal{Z}_\epsilon + \mathcal{K}(E)$ and $\mathcal{Q}_{\epsilon s} = F(\mathcal{Z}_{\epsilon s})$ form an invariant pair such that $\mathcal{Z}_{\epsilon s} = \pi_1^{-1}(\mathcal{K}\tilde{E})$ and $\mathcal{Q}_{\epsilon s} = \pi_2^{-1}(\tilde{F}(\tilde{E}))$ where $(\tilde{E}, \tilde{F})$ is the quotient pair associated with $(\mathcal{Z}_\epsilon, \mathcal{Q}_\epsilon)$. The restriction pencil associated with this pair has only trivial zeros at infinity. The dimension of $\mathcal{Z}_{\epsilon s}$ is equal to to the number of Jordan blocks at infinity plus $\sum_i (\epsilon_i + 1)$ and the dimension of $\mathcal{Q}_{\epsilon s}$ is equal to to the number of Jordan blocks at infinity plus $\sum_i \epsilon_i$. The restriction pencil associated with this pair has only trivial zeros at infinity.*

$\mathcal{Z}_{\epsilon d} = \mathcal{Z}_{\epsilon\infty} \cap F^{-1}(E)$ and $\mathcal{Q}_{\epsilon d} = F(\mathcal{Z}_{\epsilon d})$ also form an invariant pair such that $\mathcal{Z}_{\epsilon d} = \pi_1^{-1}(\tilde{F}^{-1}(\tilde{E}))$ and $\mathcal{Q}_{\epsilon d} = \mathcal{R}(\tilde{E})$. The restriction pencil associated with this pair has no trivial zeros at infinity. Dimension of $\mathcal{Z}_{\epsilon d}$ is equal to $\sum_i (\epsilon_i + 1) + \sum_j (n_j^{(\infty)} - 1)$ and dimension of $\mathcal{Q}_{\epsilon d}$ is equal to $\sum_i (\epsilon_i) + \sum_j (n_j^{(\infty)} - 1)$. The restriction pencil associated with this pair has no trivial zeros at infinity.

Proof Follows from applying lemma 2.1 and noting that $\mathcal{K}(E) \subset \mathcal{Z}_{\epsilon\infty}$, $\mathcal{Z}_{\epsilon f} \subset F^{-1}(E)$. The construction amounts to perform one step of the $\epsilon\infty$ recursion (i.e. keeping the Jordan blocks of size 1) or the ϵf recursion (i.e. removing the Jordan blocks of size 1) to the pencil $F_\infty(s)$. □

2.5. Nonorthogonal transformations

So far, the transformations that we have considered which put the pair (E, F) into block-triangular form can be realized by means of orthogonal transformations. It is possible to perform additional transformations using full row or full column matrices as pivots to zero a block row or a block column in the transformed pair. We have:

that $Q\begin{bmatrix} E & F\end{bmatrix}Z$ *takes the form:*

$$\begin{bmatrix} \begin{bmatrix} -B_\epsilon & sI-A_\epsilon \end{bmatrix} & * & * & * \\ 0 & F_\infty(s) & * & * \\ 0 & 0 & F_f(s) & * \\ 0 & 0 & 0 & \begin{bmatrix} sI-A_\eta \\ -C_\eta \end{bmatrix} \end{bmatrix} \quad (3)$$

where the pair (A_ϵ, B_ϵ) *is controllable and the pair* (A_η, C_η) *is observable.* A_ϵ *matrix (resp.* A_η*) has dimension* $\sum_i \epsilon_i$ *(resp.* $\sum_i \eta_i$*) and the column dimension of the* B_ϵ *matrix (resp.* C_η *matrix) is equal to the number of c.m.i. indices* ϵ_i*'s (resp. r.m.i. indices* η_i*'s).*

Proof Using $\overline{E}_\epsilon$, which has full row rank, as a pivot, the first row of (2) can be transformed as shown (where the terms denoted $*$ are constant). Since the pencil $F_\epsilon(s)$ has only "c.m.i" blocks and no finite zeros, the pair (A_ϵ, B) is controllable. Results concerning the η-part follows by duality. □

Pole placement

The preceding transformation can be used for finding a matrix gain F which places the (finite or infinite) controllable modes of a *regular* pencil $(sE-A)$. Consider the nonsquare "extended" pencil $P_B(s) = \begin{bmatrix} -B & sE-A \end{bmatrix}$. As shown above, it has an ϵ part which corresponds to the modes of $sE-A$ which can be shifted by a feedback BF. Let Q_ϵ and Z_ϵ be such that $Q_\epsilon P_B(s) Z_\epsilon = \begin{bmatrix} -B_\epsilon & sI-A_\epsilon \end{bmatrix}$. Since the pair (A_ϵ, B_ϵ) is controllable, we can use a *standard* pole placement algorithm for placing the eigenvalues of $A_\epsilon + B_\epsilon K_\epsilon$: this amounts to update the right transformation Z_ϵ by a right multiplication with the nonsingular matrix $\begin{bmatrix} I & K_\epsilon \\ 0 & I \end{bmatrix}$. If such a transformation is carried out, we obtain by theorem 2.4:

$$\left[\begin{array}{c} Q_\epsilon \\ \hline Q_2 \end{array}\right] \left[\begin{array}{c|c} -B & sE-A \end{array}\right] \left[\begin{array}{c|c|c} Z_{11} & Z_{12} & Z_{13} \\ \hline Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{array}\right] =$$

$$\left[\begin{array}{c|cc} -B_\epsilon & sI-\overline{A}_\epsilon & * \\ \hline 0 & 0 & s\tilde{E}-\tilde{A} \end{array}\right]$$

where $\sigma(\overline{A}_\epsilon) = \sigma(A_\epsilon + B_\epsilon K_\epsilon)$ and B_ϵ has the same column dimension as B. The pencil $s\tilde{E}-\tilde{A}$ contains only uncontrollable modes (and also in particular all the static modes of $(sE-A)$). It is easy to see that the matrix $Z_d = \begin{bmatrix} Z_{22} & Z_{23} \\ Z_{32} & Z_{33} \end{bmatrix}$ is invertible and thus there exists F such that $FZ_d = \begin{bmatrix} Z_{12} & Z_{13} \end{bmatrix}$. We obtain

$$\begin{bmatrix} sE-A-BF \end{bmatrix} = \begin{bmatrix} Q_\epsilon \\ Q_2 \end{bmatrix}^{-1} \begin{bmatrix} sI-\overline{A}_\epsilon & * \\ 0 & s\tilde{E}-\tilde{A} \end{bmatrix} Z_d^{-1}$$

the controllables modes of $(sE-A)$. This construction can be extended to pencils $(sE-A)$ which are right invertible. We can implement the following pole placement algorithm.

Algorithm F=`gppol(E,A,B,roots)`

Inputs: E, A two $n\times n$ real matrices defining a regular pencil, B a $n\times p$ real matrix and *roots* a set of desired eigenvalues.

Outputs: F, a real $p\times n$ matrix such that the controllable modes of the pencil $sE-A$ are located at *roots*.

3. Observer construction

3.1. A classical approach

By dualizing the algorithm given at the end of the preceding section, it is possible to construct a strictly proper observer for the system:

$$\Sigma(s) \quad \left\{ \begin{array}{rcl} (sE-A)x & = & Gu \\ y & = & Cx \end{array} \right.$$

where the pencil $sE-A$ is regular. Here u is a known input and y is an observation. The "extended" pencil is now $P_C(s) = \begin{bmatrix} sE-A \\ C \end{bmatrix}$. This pencil admits only finite and infinite zeros and rmi indices. The above equations can be written $P_c(s)x = G^{(e)}\left(\begin{bmatrix} u \\ y \end{bmatrix}\right)$ with $G^{(e)} = \begin{bmatrix} G & 0 \\ 0 & -I \end{bmatrix}$. We can construct as above two matrices Q and Z such that with appropriate partitions:

$$\left[\begin{array}{cc|c} Q_{11} & Q_{12} & Q_{13} \\ \hline Q_{21} & Q_{22} & Q_{23} \\ \hline Q_{31} & Q_{32} & Q_{33} \end{array}\right] \begin{bmatrix} sE-A \\ C \end{bmatrix} \left[\begin{array}{c|c} Z_1 & Z_\eta \end{array}\right] =$$

$$\left[\begin{array}{c|c} s\overline{E}-\overline{A} & * \\ \hline 0 & sI-A_\eta \\ \hline 0 & C_\eta \end{array}\right]$$

Here, by the regularity assumption, matrix C_η has the same row dimension as C. We may assume that, by an appropriate choice of Q matrix, the eigenvalues of A_η are set to desired values using a standard pole placement algorithm. However, the finite and infinite zeros of $s\overline{E}-\overline{A}$ are unobservable. The rhs matrix $G^{(e)}$ is updated as follows:

$$\left[\begin{array}{cc|c} Q_{11} & Q_{12} & Q_{13} \\ \hline Q_{21} & Q_{22} & Q_{23} \\ \hline Q_{31} & Q_{32} & Q_{33} \end{array}\right] \begin{bmatrix} G & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} * \\ \begin{bmatrix} G_{\eta,1} \\ G_{\eta,2} \end{bmatrix} \end{bmatrix}$$

We can now use the classical construction [8] for defining an observer for $z = Hx$. We recall that an observer

$$O(s)\quad \begin{cases} \dot{w} &= A_o w + B_o \begin{bmatrix} u \\ y \end{bmatrix} \\ \hat{z} &= C_o w \end{cases}$$

such that when t goes to infinity $\hat{z}(t) - z(t) \to 0$ for all $w(0)$.

Let $A_o = A_\eta = \begin{bmatrix} 0 & I & 0 \end{bmatrix} QAZ \begin{bmatrix} 0 \\ I \end{bmatrix}$, $B_o = \begin{bmatrix} 0 & I & 0 \end{bmatrix} QG^{(e)}$, and $H \begin{bmatrix} Z_1 \mid Z_\eta \end{bmatrix} = \begin{bmatrix} C_{\overline{o}} \mid C_o \end{bmatrix}$. From standard observer theory we know that a necessary and sufficient condition for $\hat{z}(t) \to z(t)$ with arbitrary dynamics is $C_{\overline{o}} = 0$. If this is the case, the transfer function of $u \to z$ in $\Sigma(s)$ is the same as the composed transfer function $u \to (u, y(u)) \to \hat{z}$ (we denote $y(u)$ the ouput of $\Sigma(s)$ associated with the input u). This construction is that of a standard observer, the only difference being that the outputs of $\Sigma(s)$ can be improper.

3.2. Generalized observer

We consider, in a discrete time setting, a sequence of variables $\xi(k) \quad k = 0, 1, \ldots$ linked by:

$$E\xi(k+1) - F\xi(k) = Gu(k), \quad k = 0, 1, \ldots \tag{4}$$

We assume here that all the known quantities, inputs and observations are put in the rhs member and we not not assume regularity of the pencil (E, F). Note that, in general, the sequence ξ exists only for particular inputs and particular "initial" state $\xi(0)$.

Our objective is to construct a proper (causal) observer for $z = H\xi$. From the preceding results we observe that $\tilde{\xi}(k) = \xi(k)(mod \mathcal{Z}_\epsilon)$ is uniquely defined and satisfies:

$$\tilde{E}\tilde{\xi}(k+1) = \tilde{F}\tilde{\xi}(k) + \tilde{G}u(k)$$

where $(\tilde{E}, \tilde{F})$ is the quotient pencil associated with the invariant pair $(\mathcal{Z}_\epsilon, \mathcal{Q}_\epsilon)$ and $\tilde{G} = G(mod \mathcal{Q}_\epsilon)$. In particular, we see that the sequence $z(k) = H\xi(k)$, $k = 0, 1, \ldots$, is uniquely defined iff $\mathcal{K}(H) \supset \mathcal{Z}_\epsilon$.

There exist two bases of $\mathcal{Y}$ and $\mathcal{X}$ represented by matrices Q^{-1} and Z such that $Q(sE - F)Z = \ldots$

$$\begin{bmatrix} F_{\epsilon d}(s) & * & * & * & * \\ 0 & F_b(s) & * & * & * \\ 0 & 0 & F_s & * & * \\ 0 & 0 & 0 & F_g(s) & * \\ 0 & 0 & 0 & 0 & \begin{bmatrix} sI - A_\eta \\ -C_\eta \end{bmatrix} \end{bmatrix}$$

and

$$QG = \begin{bmatrix} G_{\epsilon d} \\ G_b \\ G_s \\ G_g \\ \begin{bmatrix} G_\eta^1 \\ G_\eta^2 \end{bmatrix} \end{bmatrix}$$

$F_g(s)$ (resp. $F_b(s)$) contains all the stable (resp. unstable) finite zeros of the $sE - F$ and the pair (A_η, C_η) is observable. We have:

Theorem 3.1 *A causal (resp. strictly causal) observer can be constructed for $z(k) = H\xi(k)$ iff $\mathcal{K}(H) \supset Z_{\epsilon d} + Z_{\epsilon b}$ (resp. $\mathcal{K}(H) \supset Z_{\epsilon\infty} + Z_{\epsilon b}$). An observer with arbitrary error dynamics can be constructed for $z(k) = H\xi(k)$ iff $\mathcal{K}(H) \supset Z_{\epsilon f\infty}$*

Proof Only if: Make a realization for the error dynamics from $\Sigma(s)$ and $O(s)$ and conclude that if e goes to zero necessarily $C_{\overline{o}} = 0$. This mean $\mathcal{K}(H) \supset$ the appropriate subspace.

If: constructive. By the previous lemma we know for each $\mathcal{Z}$ the dynamics of $x(mod \mathcal{Z})$ and we take this as an observer.

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